Multivariable Rational Matrix Functions: Synthesis Constraints Using Reciprocal Electrical Networks

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Two-element kind networks

- $G = (V, E)$ – finite linear digraph with fixed ordering of its nodes V and edges E.
- $\nabla \in \mathbb{C}^{|E|\times|V|}$ – matrix representation for its discrete gradient*:

$$(\nabla u)_\ell = u_i - u_j,\ u \in \mathbb{C}^{|V|},\ \ell = 1, \dots, |E|,$$

$$v_i \xrightarrow{e_\ell} v_j$$

- $\sigma : \mathbb{C}^2 \to \mathbb{C}^{|E|\times|E|}$ is "two-element-kind" edge function:

$$\sigma(z_1, z_2) = z_1 \mathrm{diag}(\kappa) + z_2 \mathrm{diag}(\mu),\ \kappa, \mu \in [0, \infty)^{|E|}.$$

- $L : \mathbb{C}^2 \to \mathbb{C}^{|V|\times|V|}$ is the Laplacian matrix for the network:

$$L = \nabla^T \sigma \nabla = z_1 K + z_2 M, \tag{1}$$
$$K = \nabla^T \mathrm{diag}(\kappa)\nabla,\ M = \nabla^T \mathrm{diag}(\mu)\nabla.$$

- Physical viewpoint: Edge e_ℓ is a parallel connection of a $K-$type element with value $z_1\kappa_\ell$ with a $M-$type element with value $z_2\mu_\ell$.

* ∇^T = complete node-edge incidence matrix of G, cf. N. Balabanian & T. Bickart. *Electrical Network Theory*. Wiley, 1969.

Dirichlet-to-Neumann (DtN) map of the network

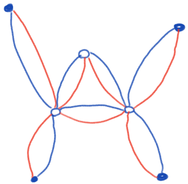

Figure: Example 2-element-kind network with 4 boundary nodes (•) & 3 interior nodes (∘). The blue (resp. red) edges are $K-$type (resp. $M-$type) elements.

- Nodes V partitioned into boundary nodes B and interior nodes I:

$$V = B \cup I, B \cap I = \emptyset.$$

- Dirichlet problem: Find $u \in \mathbb{C}^{|V|}$, given $f \in \mathbb{C}^{|B|}$, such that

$$(Lu)_I = 0, \; u_B = f. \tag{2}$$

Lemma 1 (DtN map is well-defined)

If $L_{II} \in \mathbb{C}^{|I| \times |I|}$ is invertible, then (2) admits a unique solution $u \in \mathbb{C}^{|V|}$ for each $f \in \mathbb{C}^{|B|}$ and the response matrix $W \in \mathbb{C}^{|B| \times |B|}$ is well-defined by

$$Wf = (Lu)_B. \tag{3}$$

Electric circuit interpretation

- For an all-terminal network (i.e., $|V| = |B|$), the response matrix is $W = L$.
- From now on, assume $\exists (z_1, z_2) \in \mathbb{C}^2$ such that L_{II} is invertible, if $|V| \neq |B|$.
  - e.g., true if G is a connected graph and $\kappa_j + \mu_j \neq 0$ for all j.
- The matrix-valued function $W(z_1, z_2)$ is the Dirichlet-to-Neumann (DtN) map (response matrix) of the network $(G, \sigma(z_1, z_2))$.
- When $z_1, z_2 > 0$, it is the voltage to current map at the boundary nodes B for an R circuit (i.e., resistor only network) with edge conductances $\sigma(z_1, z_2)$.
- If we let z_1, z_2 take complex values, the response also has an interpretation as a two-element-kind network such as RC, RL, or LC circuit.
  - E.g., $W(z_1, z_2)$ with $z_2 = \mathrm{i}\omega$ and $z_1, \omega > 0$ is the response matrix of an RC circuit at frequency ω with edge conductances $z_1 \kappa$ and edge capacitances μ.

Kirchhoff matrices and their circuit realizations

Definition 2 (Kirchhoff matrix)

We call a matrix $A \in \mathbb{R}^{n \times n}$ *a Kirchhoff matrix if* $A_{ij} \leq 0$ *for all* $i \neq j$*,* $A^T = A$*, and* $A1 = 0$*, where* $1 \in \mathbb{R}^n$ *is the vector of all ones.*

Proposition 3 (Realization of a Kirchhoff matrix*)

The following statements are equivalent:

(i) *A is an $n \times n$ Kirchhoff matrix.*

(ii) *A is a response matrix of a resistor only network with n boundary nodes.*

(iii) *A is a response matrix of a resistor only, all-terminal network with n boundary nodes.*

* E. Curtis and J. Morrow. *Inverse Problems For Electrical Networks*. World Scientific, 2000.

Structured matrices associated with Kirchhoff matrices

Definition 4 (Z-, M-matrix)

A Z-matrix is a real square matrix whose off-diagonal entries are less than or equal to zero. An M-matrix is a Z-matrix whose eigenvalues have nonnegative real parts (e.g., a Kirchhoff matrix).

Definition 5 (Stieltjes matrix)

A Stieltjes matrix is symmetric M-matrix that is invertible (e.g., any invertible principal submatrix of a Kirchhoff matrix).

Definition 6 (Row-stochastic matrix)

A real matrix is called row-stochastic if it has all entries nonnegative such that the entries in each row sum to 1.

How do these structured matrices arise in circuit theory?

This lemma is well-known[*] which can prove the next proposition.

Lemma 7 (Inverses of *M*- and Stieltjes matrices)

The inverse of an invertible M-matrix is a matrix whose entries are nonnegative. Moreover, the inverse of a Stieltjes matrix is a real symmetric positive definite matrix whose entries are all nonnegative.

Proposition 8 (Structured matrices arising in R circuits)

Let L be a Kirchhoff matrix of a resistor only network with boundary and interior nodes B and I, respectively, such that L_{II} is invertible. Then the network response matrix W is the Schur complement

$$W = L_{BB} - L_{BI} L_{II}^{-1} L_{IB}. \tag{4}$$

Moreover, L_{II} is a Stieltjes matrix. Furthermore, both L_{BB} and $L_{BI} L_{II}^{-1} L_{IB}$ are real symmetric positive semidefinite matrices such that the off-diagonal entries are nonpositive and nonnegative, respectively, (in particular, L_{BB} is an M-matrix) and $-L_{II}^{-1} L_{IB}$ is a row-stochastic matrix.

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[*] A. Berman & R. J. Plemmons. *Nonnegative Matrices in the Mathematical Sciences*. SIAM, 1994.

[*] C. R. Johnson, R. L. Smith, & M. J. Tsatsomeros. *Matrix Positivity*. Cambridge Univ. Press, 2020.

Structured matrices for two-element-kind networks?

[1] F. Guevara Vasquez, G. W. Milton, & A. Welters. *Realization of arbitrary resonances in two-element-kind networks*. In preparation.

Theorem 9 (Guevara Vasquez, Milton, Welters, 2025)

The response matrix $W = W(z_1, z_2)$ has the form:

$$W(z_1, z_2) = z_1 A + z_2 B + \sum_{l=1}^{N} \frac{C^{(l)}}{\alpha_1^{(l)}/z_1 + \alpha_2^{(l)}/z_2} \tag{5}$$

where

i) $\alpha_1^{(l)} + \alpha_2^{(l)} = 1$ *and* $\alpha_1^{(l)}, \alpha_2^{(l)} > 0$*;*

ii) *A, B are $|B| \times |B|$ Kirchhoff matrices;*

iii) *$C^{(l)}$ are real symmetric positive semidefinite $|B| \times |B|$ matrices satisfying $C^{(l)}1 = 0$.*

iv) *$W(z_1, z_2)$ is a $|B| \times |B|$ Kirchhoff matrix, for every $z_1, z_2 > 0$.*

Proof via analogy with theory of composites

[2] G. W. Milton (Ed.), *Extending the Theory of Composites to Other Areas of Science*. Milton-Patton Publishing, 2016.

[3] G. W. Milton. *The Theory of Composites*. Classics in Applied Mathematics, SIAM, 2022.

Proof.

The representation (1) for $L = L(z_1, z_2)$ we conclude $W = W(z_1, z_2)$ is a:

(I) rational $\mathbb{C}^{|B|\times|B|}$-valued function of the two complex variables z_1, z_2;

(II) homogeneous function of degree one, i.e., $W(\lambda z_1, \lambda z_2) = \lambda W(z_1, z_2)$ for $\lambda \in \mathbb{C} \setminus \{0\}$.

(III) Herglotz function, i.e., $\operatorname{Im} W(z_1, z_2) \geq 0$ if $\operatorname{Im} z_1 > 0$ and $\operatorname{Im} z_2 > 0$.

By well-known results (see [2,3]) it has the decomposition (5) with property i) such that A, B, and $C^{(l)}$ are real symmetric positive semidefinite $|B| \times |B|$ matrices. The remaining properties ii) and iii) in Theorem 9 are an immediate consequence of the fact [which proves iv)] that $W(z_1, z_2)$ is a Kirchhoff matrix whenever $z_1 > 0$ and $z_2 > 0$ since this is true of $L(z_1, z_2)$ by (1). $\square$

Realization problem is unsolved, but [1] is progress

- $(\alpha_1^{(l)}, \alpha_2^{(l)})$, $C^{(l)}$ – a "finite pole" of W and its associated "residue".
- A (resp. B) – the "residue" at the "infinite pole" $(1, 0)$ [resp. $(0, 1)$].
- *Realization problem*: given residues $C^{(l)}$ and poles $(\alpha_1^{(j)}, \alpha_2^{(l)})$ satisfying properties i) and iii) from Theorem 9, can we find conditions on A and B that guarantee $W(z_1, z_2)$ in (5) is the response of a two-element-kind network.
- Realization problem is an open problem.
  - Why? Partially answer, all constraints on the poles and structure of the residues is not completely known.
- Main results of [1] is to provide some progress on this realization problem.

Networks with one interior node: Star networks

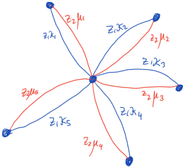

Figure: A star network with $n = 5$ boundary nodes ($\bullet$) and one interior node ($\circ$).

- Star network (e.g., Fig. 2): $|B| = n$, $|I| = 1$.
- $\kappa, \mu \in [0, \infty)^n$ with nonzero $k = \kappa^T 1$ and $m = \mu^T 1$.
- $\mathrm{star}(\kappa) = \mathrm{diag}(\kappa) - \kappa\kappa^T / 1^T\kappa$ – response matrix of a star network with n edges and edge conductances κ.

A parallel sum on each $j-$th edge of $z_1\kappa_j$ and $z_2\mu_j$ gives a combined response matrix with a single finite pole and rank one residue:

$$W(z_1, z_2) = \mathrm{star}(z_1\kappa + z_2\mu) = z_1\mathrm{star}(\kappa) + z_2\mathrm{star}(\mu) + \frac{cc^T}{\alpha_1/z_1 + \alpha_2/z_2}, \tag{6}$$

$$\alpha_1 = \frac{k^{-1}}{k^{-1} + m^{-1}},\ \alpha_2 = \frac{m^{-1}}{k^{-1} + m^{-1}}, \tag{7}$$

$$cc^T = \frac{1}{k^{-1} + m^{-1}} \left(\frac{\kappa}{k} - \frac{\mu}{m}\right) \left(\frac{\kappa}{k} - \frac{\mu}{m}\right)^T. \tag{8}$$

Networks with one interior node

Theorem 10 (Guevara Vasquez, Milton, Welters, 2025)

Let $c \in \mathbb{R}^n$ be nonzero such that $c^T 1 = 0$ and $\alpha_1, \alpha_2 > 0$ with $\alpha_1 + \alpha_2 = 1$ be given. Then for any Kirchhoff $n \times n$ matrices A and B, we can realize a response matrix with a single finite pole (α_1, α_2) and rank one residue cc^T of the form

$$W(z_1, z_2) = z_1 \left[star\left(\frac{1^T a}{\alpha_1} a \right) + A \right] + z_2 \left[star\left(\frac{1^T b}{\alpha_2} b \right) + B \right] + \frac{cc^T}{\alpha_1/z_1 + \alpha_2/z_2}, \quad (9)$$

for any $a, b \geq 0$ satisfying either $c = \pm(a - b)$. In particular, this response can be obtained by "parallel sum" of an all-terminal network with response $z_1 A + z_2 B$ and a star network with response $star\left(z_1 \frac{1^T a}{\alpha_1} a + z_2 \frac{1^T b}{\alpha_2} b \right)$.

Note: One particular choice for either $c = \pm(a - b)$ is $a = c_+ = \max(c, 0)$ and $b = c_- = \max(-c, 0)$.

Main Result: Realizations of finite poles and residues

The following is the main result of our paper [1]:

Theorem 11 (Guevara Vasquez, Milton, Welters, 2025)

If $N = 0, 1, 2 \dots$ and for each $l = 1, \dots, N$, $C^{(l)}$ is a real symmetric positive semidefinite $n \times n$ matrix satisfying $C^{(l)}1 = 0$, and $\alpha_1^{(l)}, \alpha_2^{(l)}$ are scalars satisfying $\alpha_1^{(l)} + \alpha_2^{(l)} = 1$, and $\alpha_1^{(l)}, \alpha_2^{(l)} > 0$ then there exists a network with response matrix

$$W(z_1, z_2) = z_1 A + z_2 B + \sum_{l=1}^{N} \frac{C^{(l)}}{\alpha_1^{(l)}/z_1 + \alpha_2^{(l)}/z_2}, \tag{10}$$

for some Kirchhoff matrices A and B. In particular, it can be formed by a parallel sum of star networks and an all-terminal network.

Proof of main result

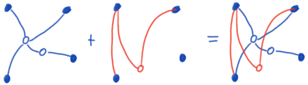

Figure: A parallel sum of the networks in red and blue corresponds to the network where the corresponding boundary nodes are joined together.

- Parallel sum of a network corresponds to the network obtained by joining the boundary nodes of several networks, where the correspondence between boundary nodes follows the boundary node ordering used for the original response matrix (see Fig.).

Proof.

First, using the spectral decomposition of $C^{(l)}$, we can write it as a finite sum of real symmetric positive semidefinite rank one matrices with the vector 1 in their nullspace. Second, by Theorem 10, there is a network (formed by a parallel sum of an all-terminal network and a star network) whose response matrix has any rank one matrix as a residue at a desired finite pole location. The theorem now follows from these two facts and using parallel sums (see Lemma 12). □

Conclusion

- Our main result (Theorem 11) is only a step towards the synthesis of two-element-kind networks (i.e., toward solving the open realization problem).
  - Doesn't reveal the dependency of A and B on the finite poles and residues: $(\alpha_1^{(l)}, \alpha_2^{(l)}), C^{(l)}$.
  - What is their interdependent structure?
  - In contrast, completely understood for single-element-kind networks, e.g., Kirchhoff matrices for R circuits.
- However, our result does show a surprisingly large class of responses can be realized with two-element-kind networks.
- Can the (abstract) theory of composites help solve the realization problem? See [2,3] and

[4] K. Beard, A. Stefan, R. Viator, & A. Welters. *Effective operators and their variational principles for discrete electrical network problems*. J. Math. Phy. 64, 073501 (2023).

- Can one use or develop fast algorithms that exploit matrix structure to help in computational problems for two-element-kind networks involving their response matrices?

Auxiliary Slides

Relevant references

Main references:

[1] F. Guevara Vasquez, G. W. Milton, & A. Welters. *Realization of arbitrary resonances in two-element-kind networks*. In preparation.

[2] G. W. Milton (Ed.), *Extending the Theory of Composites to Other Areas of Science*. Milton-Patton Publishing, 2016.

[3] G. W. Milton. *The Theory of Composites*. Classics in Applied Mathematics, SIAM, 2022.

[4] K. Beard, A. Stefan, R. Viator, & A. Welters. *Effective operators and their variational principles for discrete electrical network problems*. J. Math. Phy. 64, 073501 (2023).

Additional references:

[5] O. Wing. *Classical Circuit Theory*. Springer, 2008. Classical results for two-element-kind networks of 1-port (i.e., $|B| = 2$) networks.

[6] C. R. Johnson, R. L. Smith, & M. J. Tsatsomeros. *Matrix Positivity*. Cambridge Univ. Press, 2020. Excellent reference on known results on the structured matrices: Z-, M-, Stieltjes, and row-stochastic matrices.

Parallel sums lemma

Lemma 12 (Guevara Vasquez, Milton, Welters, 2025)

Let $W^{(j)}$ be the response matrix of a finite linear digraph $G_j = (V_j, E_j)$ with Laplacian matrix $L^{(j)}(z_1, z_2) = z_1 K^{(j)} + z_2 M^{(j)}$, boundary nodes $B = B_j$, and interior nodes I_j, for $j = 1, \ldots, p$ (with $p \geq 2$). Then the response matrix of the parallel sum of the networks is

$$W(z_1, z_2) = W^{(1)}(z_1, z_2) + \cdots + W^{(p)}(z_1, z_2), \tag{11}$$

where parallel sum of the networks is the network with boundary nodes B, interior nodes $I = I_1 \cup \ldots \cup I_p$ (the union is understood as a disjoint union), and Laplacian matrix L given blockwise by:

$$\begin{aligned} L_{BB} &= (L^{(1)})_{B_1 B_1} + \cdots + (L^{(p)})_{B_p B_p}, \\ L_{I_i I_j} &= \delta_{ij} (L^{(i)})_{I_i I_i}, \ \ i, j = 1, \ldots, p, \\ L_{B I_i} &= L_{I_i B}^T = (L^{(i)})_{B_i I_i}, \ \ i = 1, \ldots, p. \end{aligned} \tag{12}$$

The following result proved in [1], which is essentially a consequence of Foster's classic realization of 1–port or 2–terminal LC networks [5].

Theorem 13 (Realizations for 1-ports)

A function $W(z_1, z_2)$ is the response matrix of a two-element-kind network with two boundary nodes if and only if it has the form

$$W(z_1, z_2) = \left(\mathfrak{a} z_1 + \mathfrak{b} z_2 + \sum_{l=1}^{N} \frac{\mathfrak{c}^{(l)}}{\alpha_1^{(l)}/z_1 + \alpha_2^{(l)}/z_2} \right) cc^T \tag{13}$$

where $c = [1, -1]^T$ with scalars $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}^{(l)}, \alpha_1^{(l)}, \alpha_2^{(l)} \geq 0$ such that $\alpha_1^{(l)} + \alpha_2^{(l)} = 1$.

Realizations by connecting 1-ports

If we use two-terminal-networks to connect boundary nodes, we cannot realize the general class of residues that can be realized with our main result (Theorem 11). The next theorem shows that the possible residues have the additional constraint of being Kirchhoff matrices.

Theorem 14

An $n \times n$ matrix-valued function

$$W(z_1, z_2) = z_1 A + z_2 B + \sum_{l=1}^{N} \frac{C^{(l)}}{\alpha_1^{(l)}/z_1 + \alpha_2^{(l)}/z_2}, \tag{14}$$

is a response matrix of network with n boundary nodes, if

(i) $\alpha_1^{(l)} + \alpha_2^{(l)} = 1$ *and* $\alpha_1^{(l)}, \alpha_2^{(l)} > 0$;

(ii) A, B, *and* $C^{(l)}$ *are Kirchhoff matrices.*