

The Algebraic Cost of a Boolean Sum

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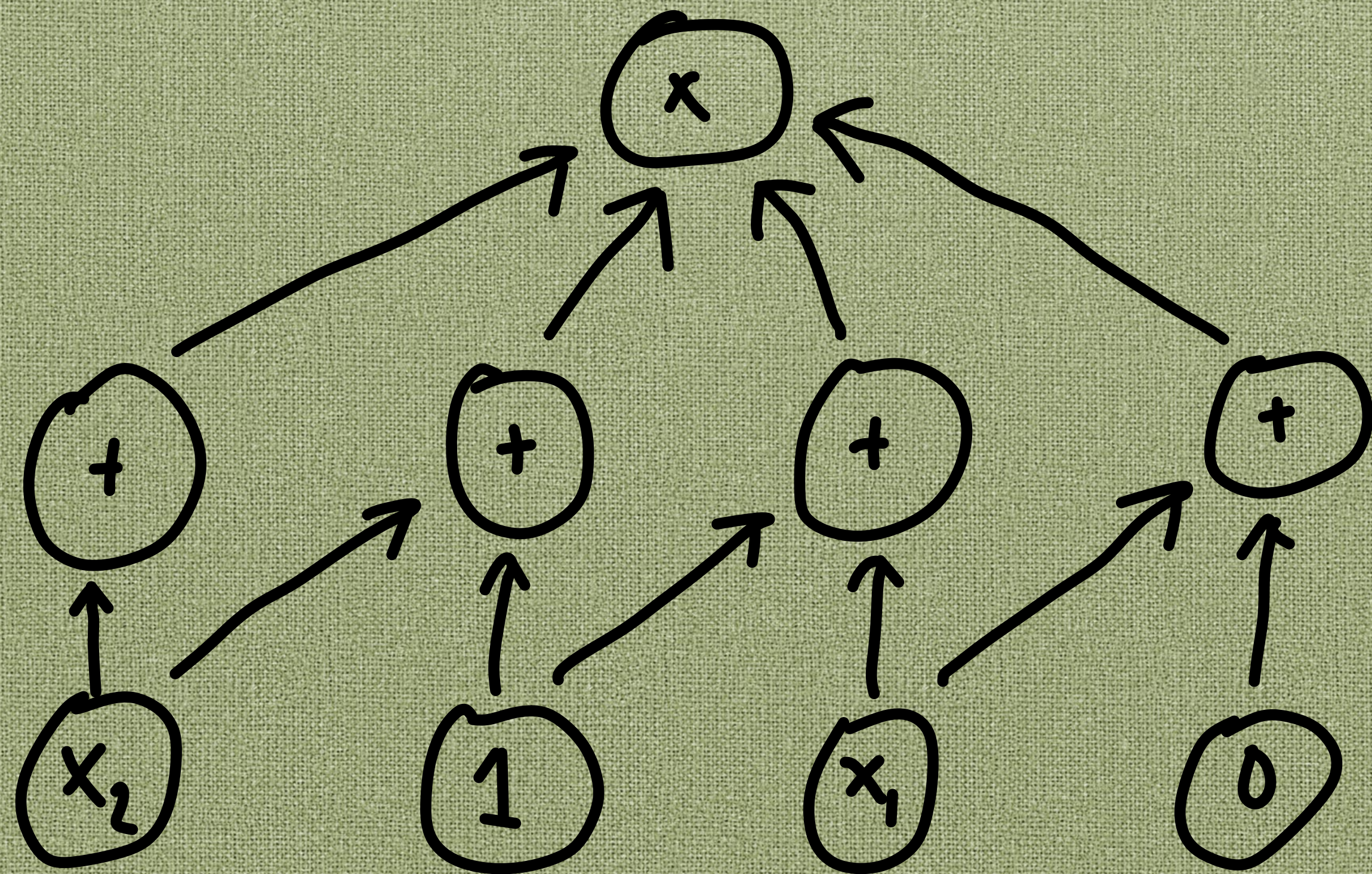
Sébastien Tavenas
[u. Savoie Mont Blanc]



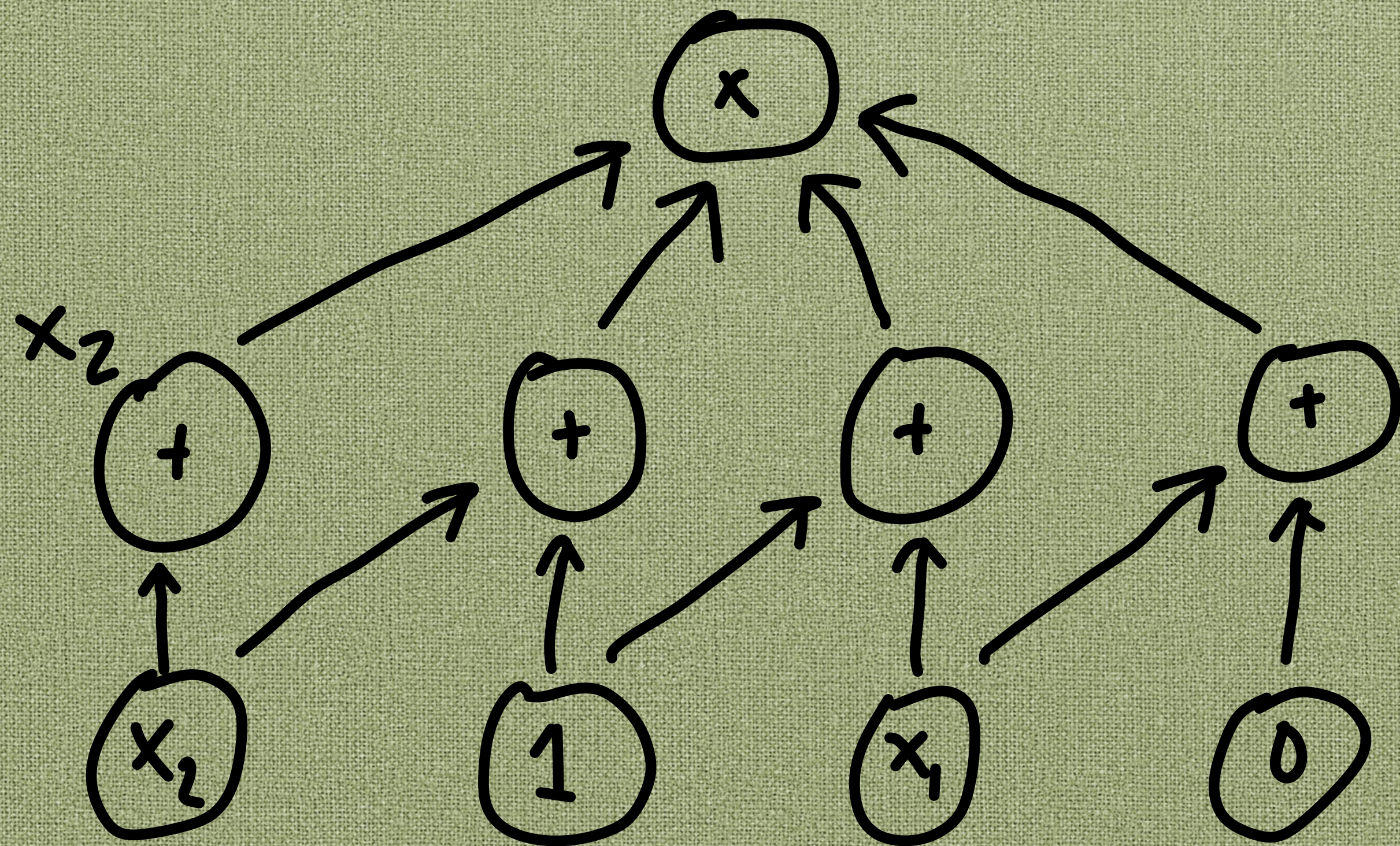
Amir Yehudayoff
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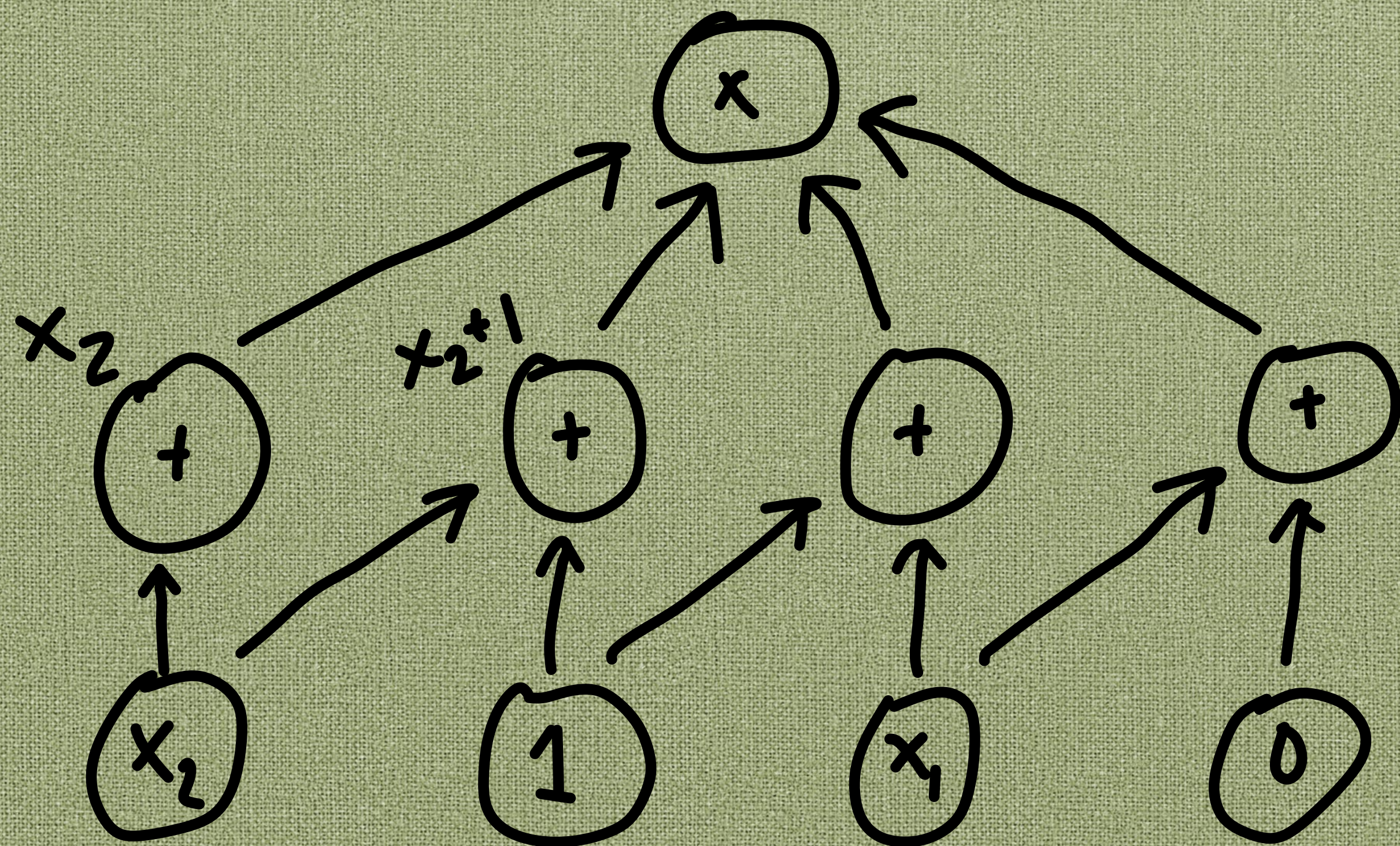
Algebraic Complexity



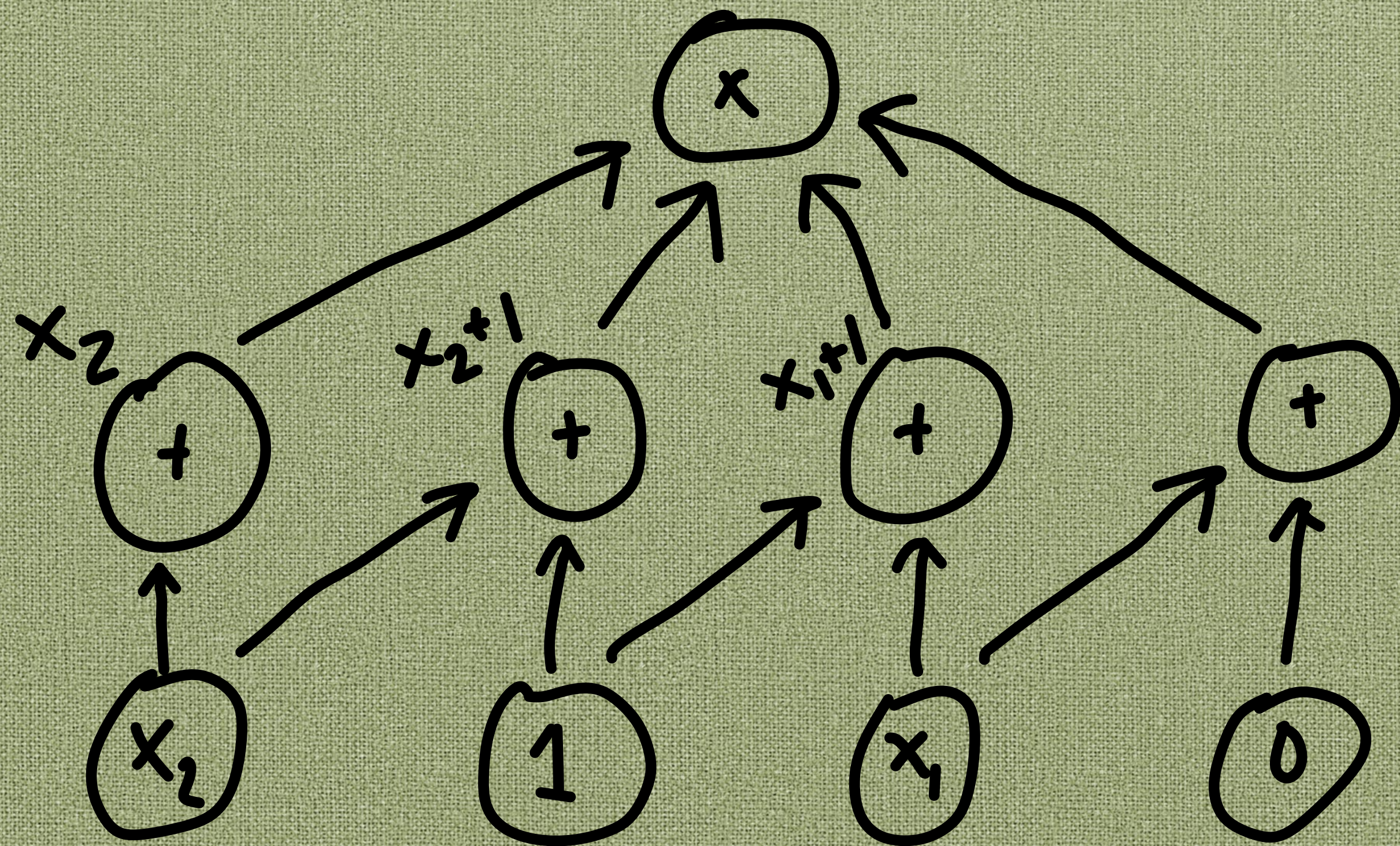
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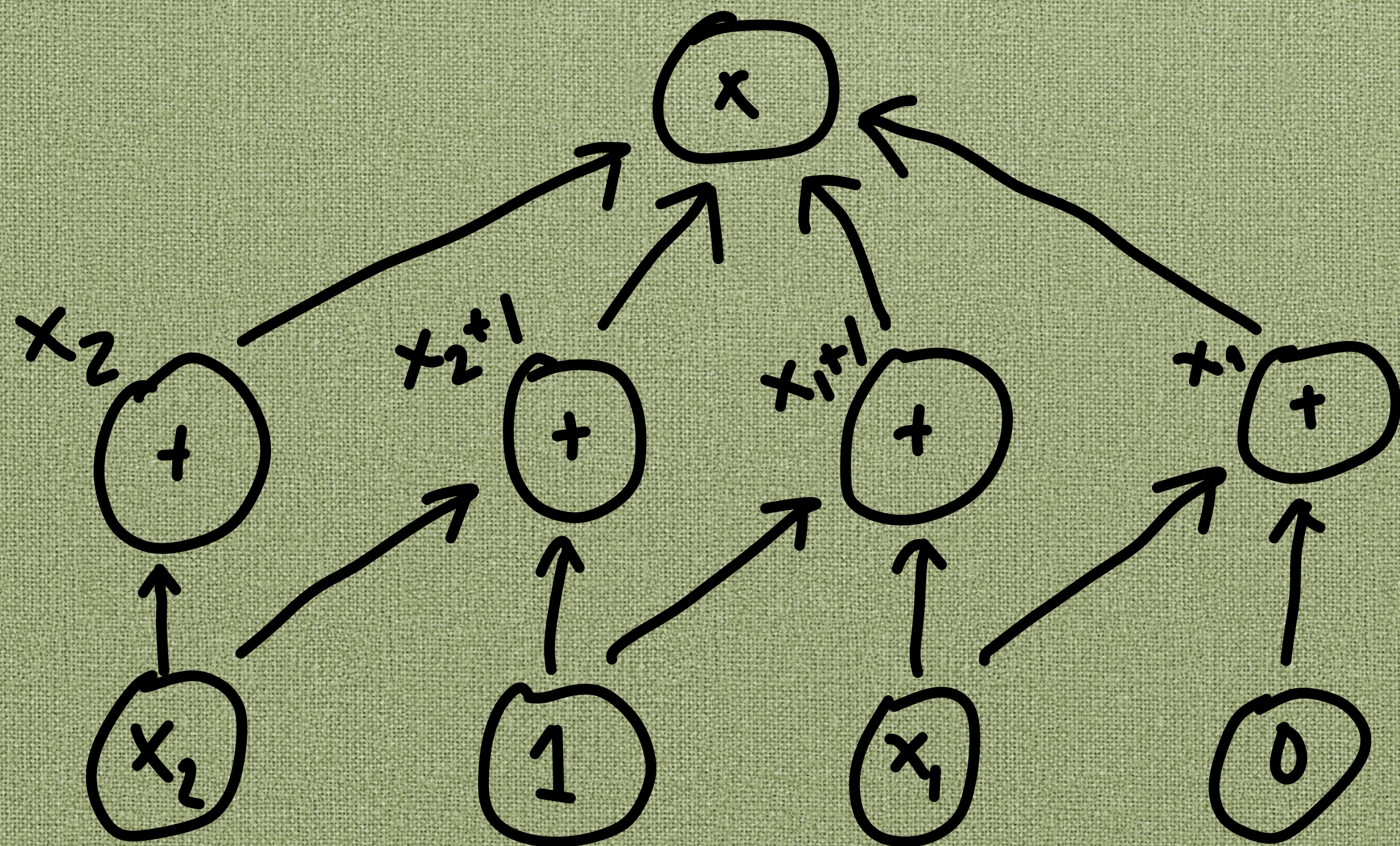
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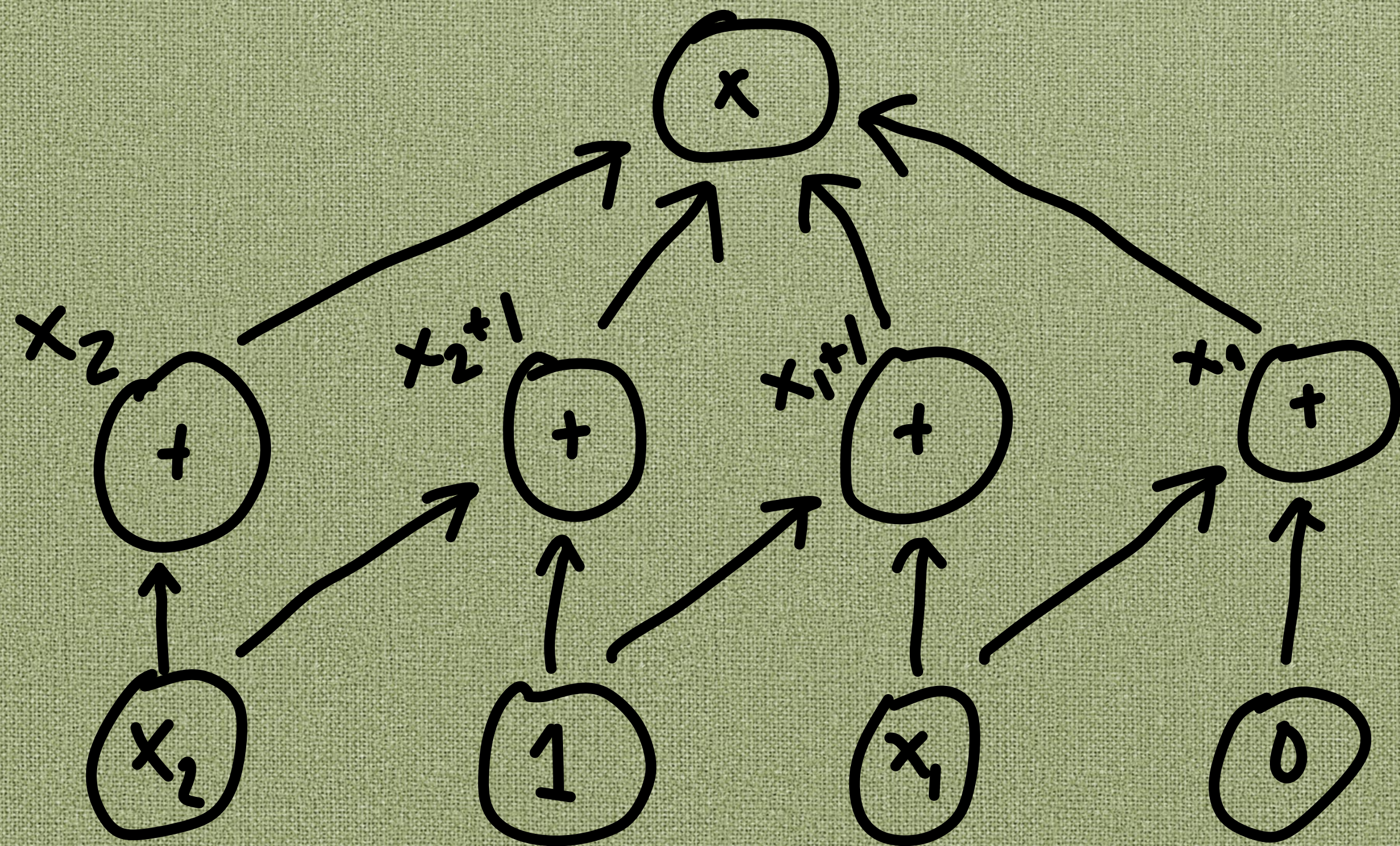


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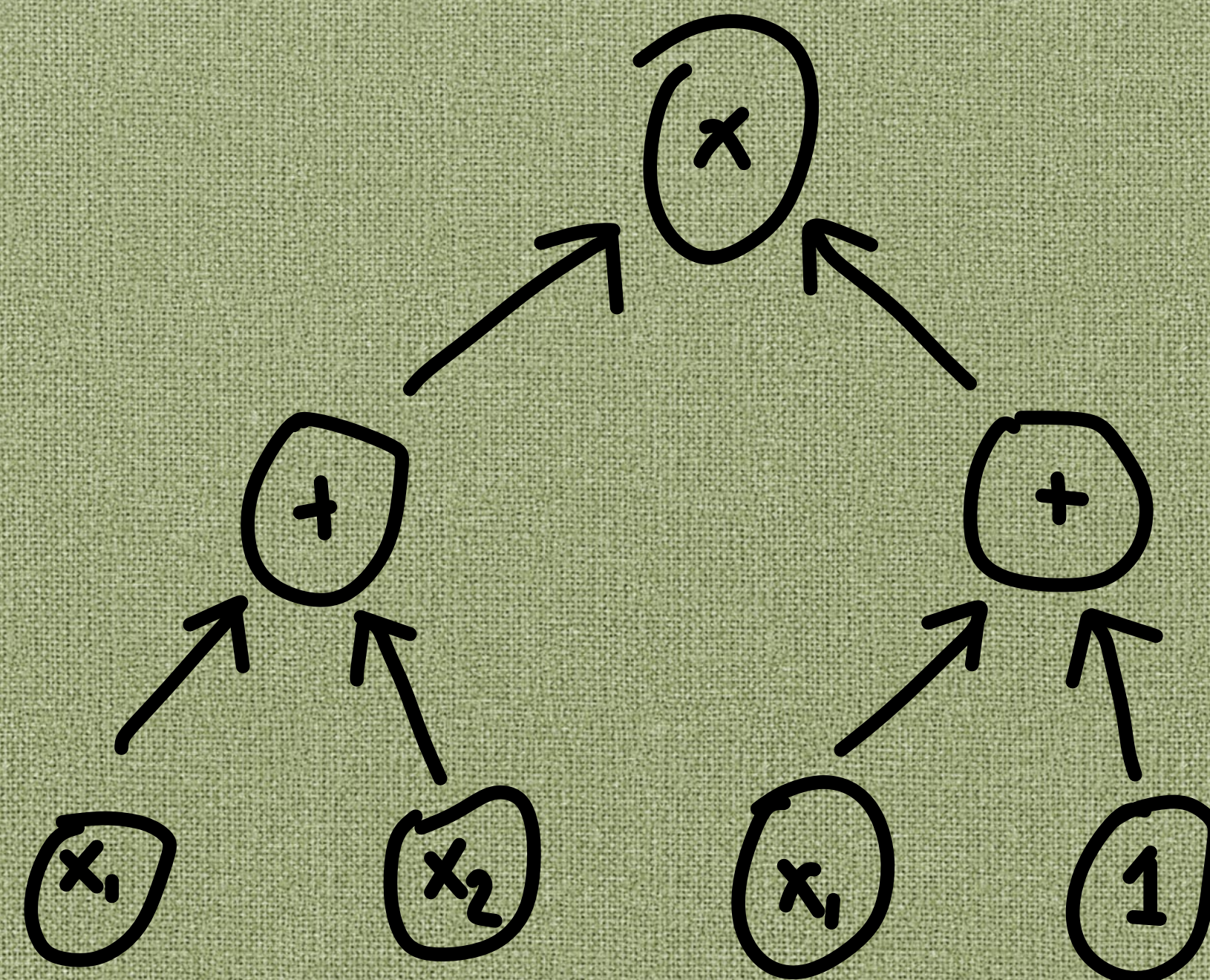
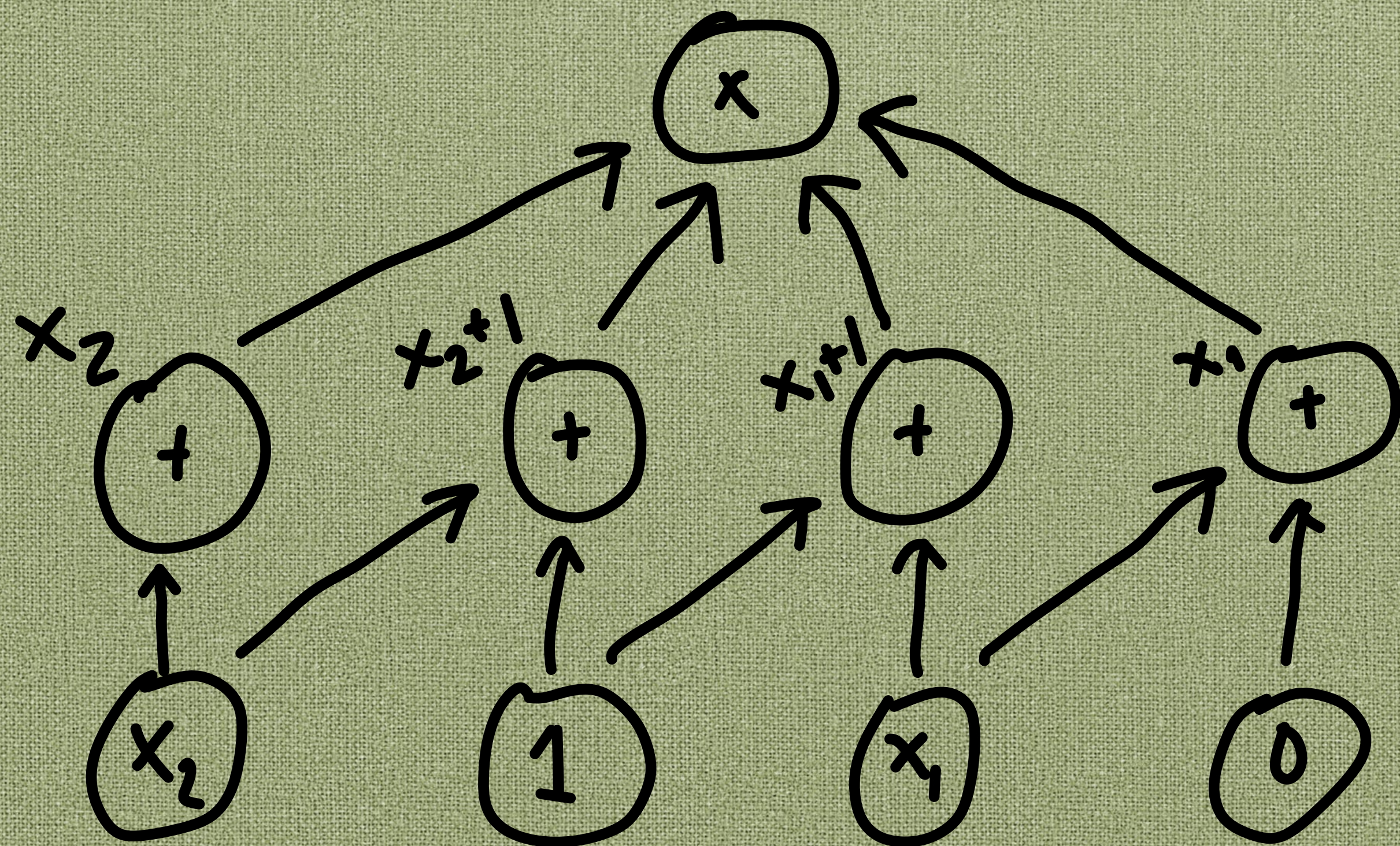
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$$x_1 \cdot (x_2 + 1) \cdot (x_1 + 1) \cdot x_1$$



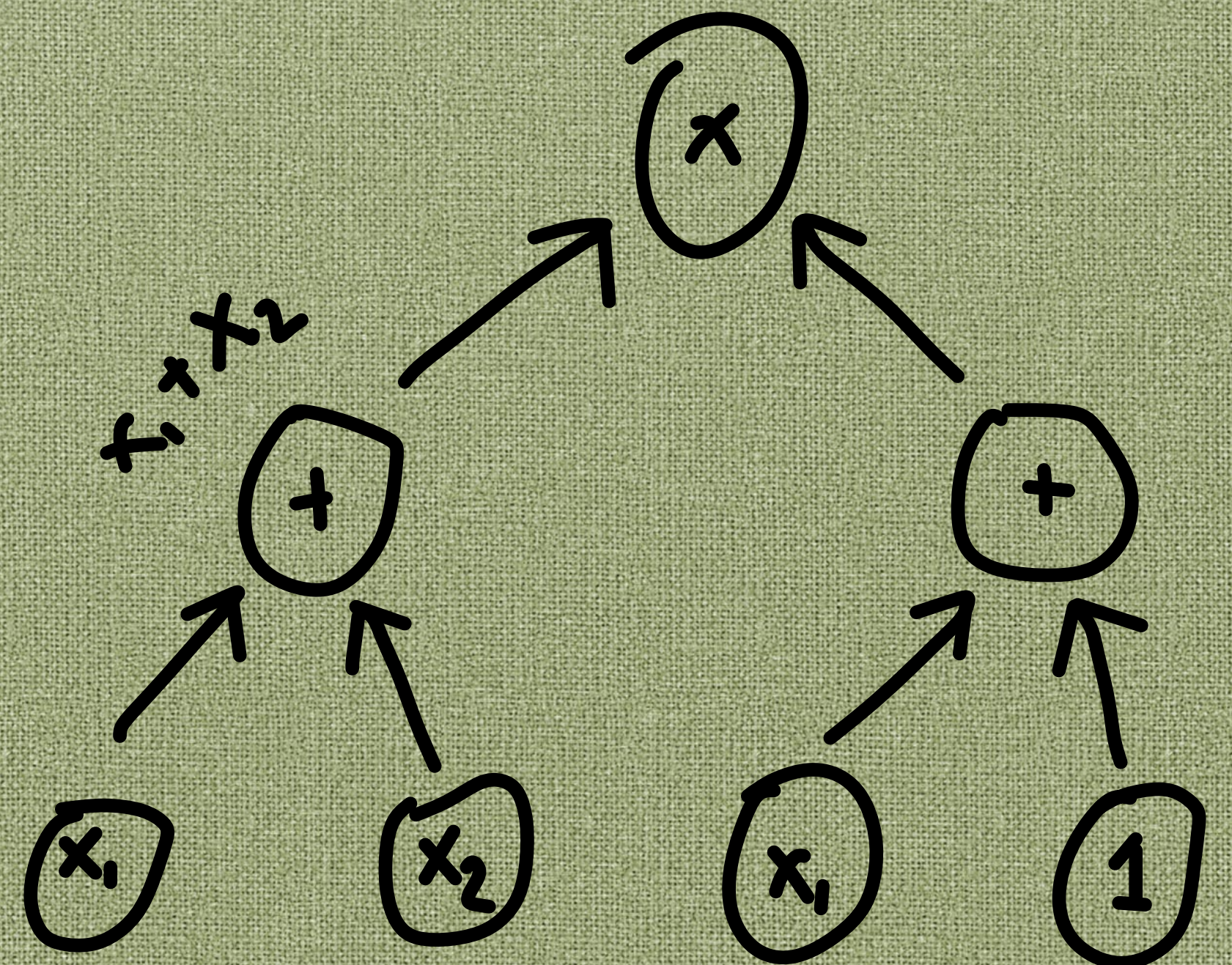
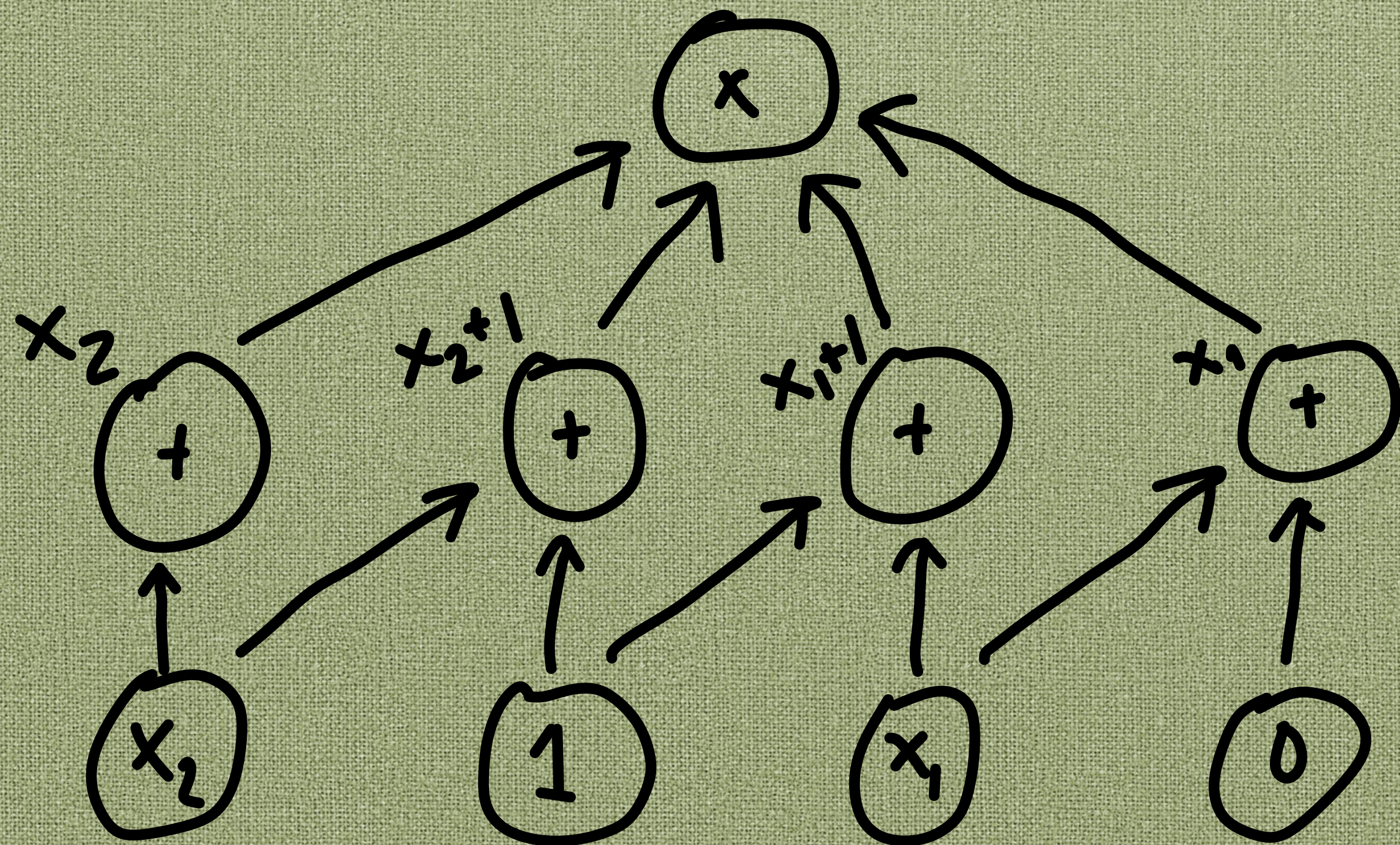
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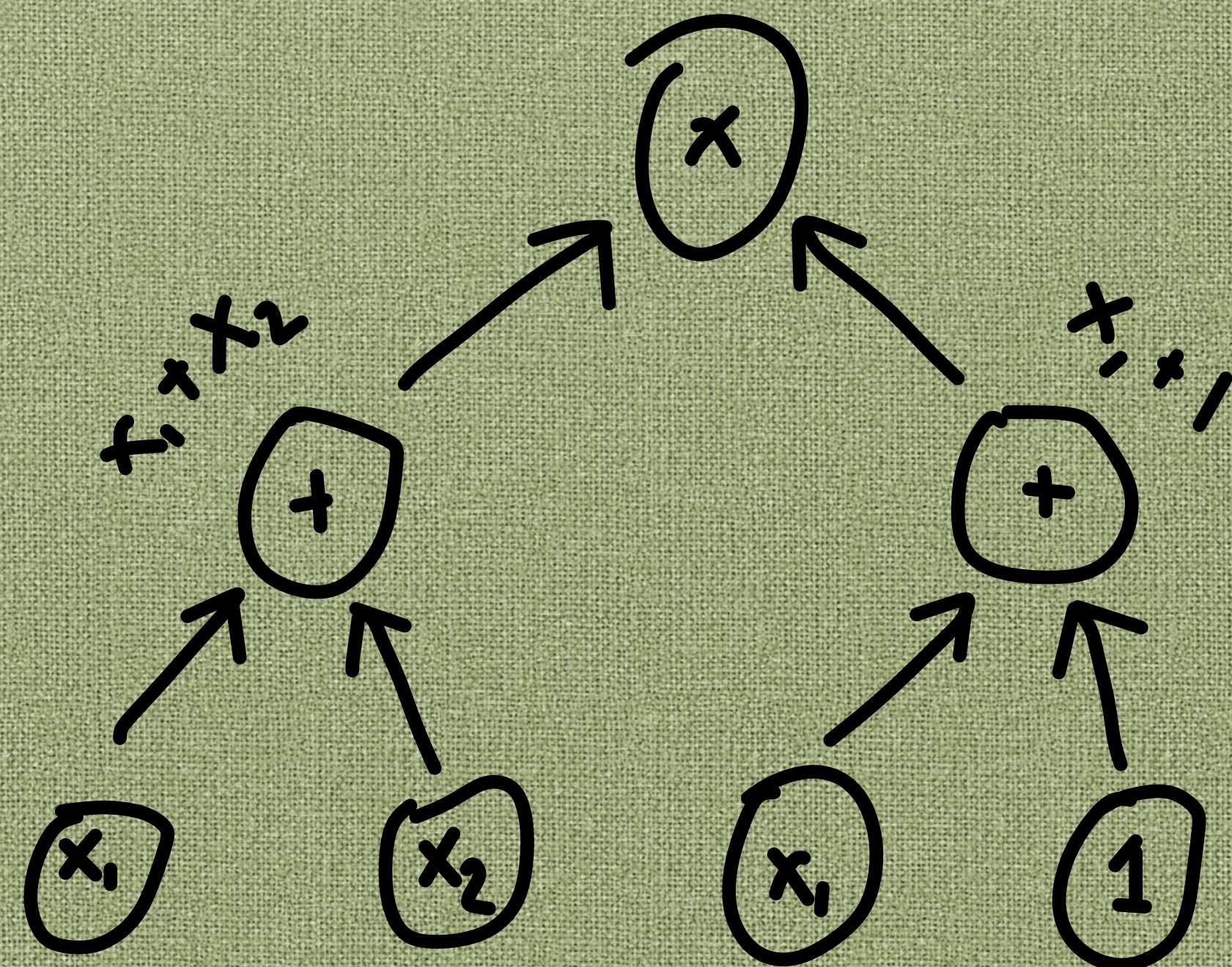
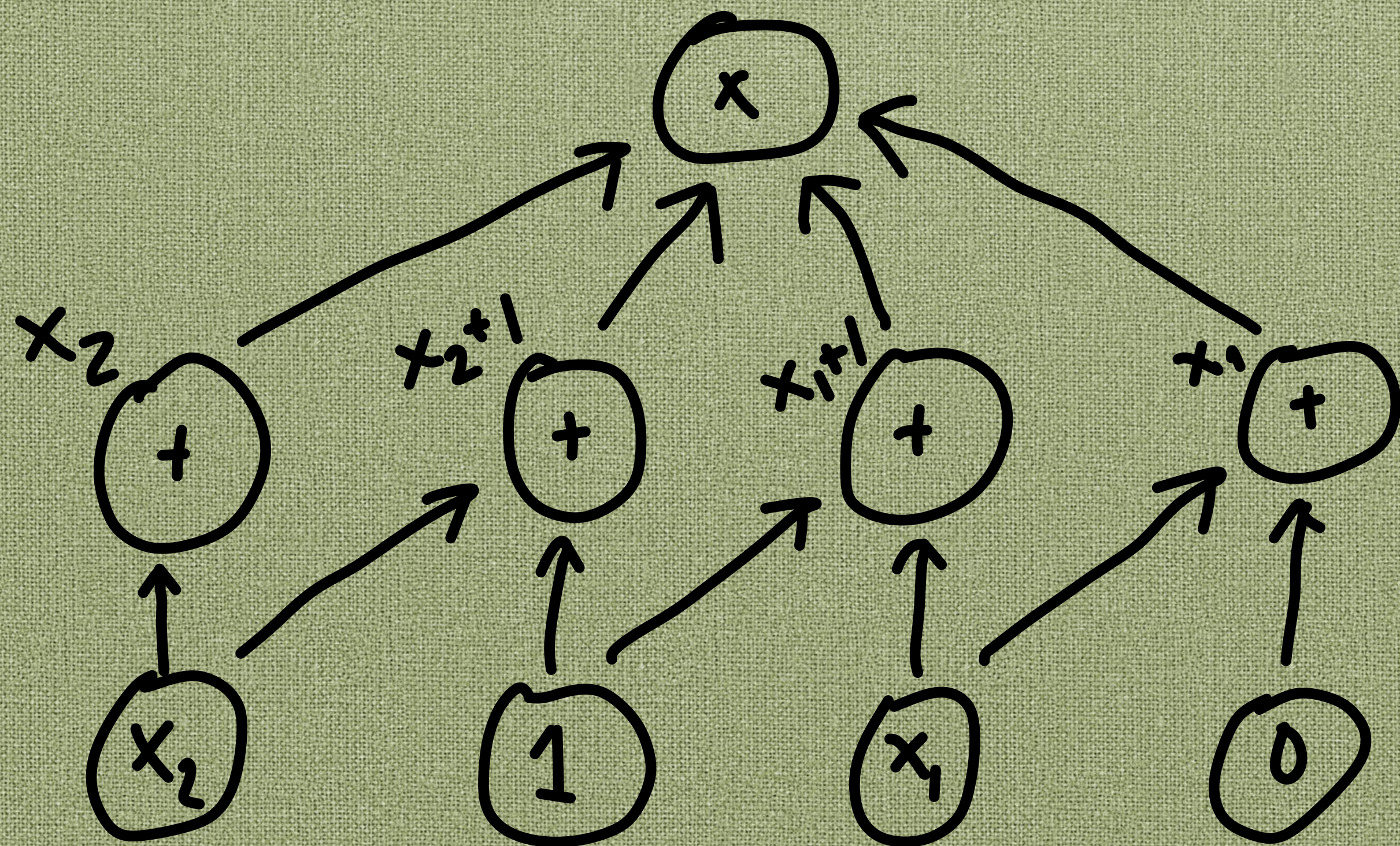
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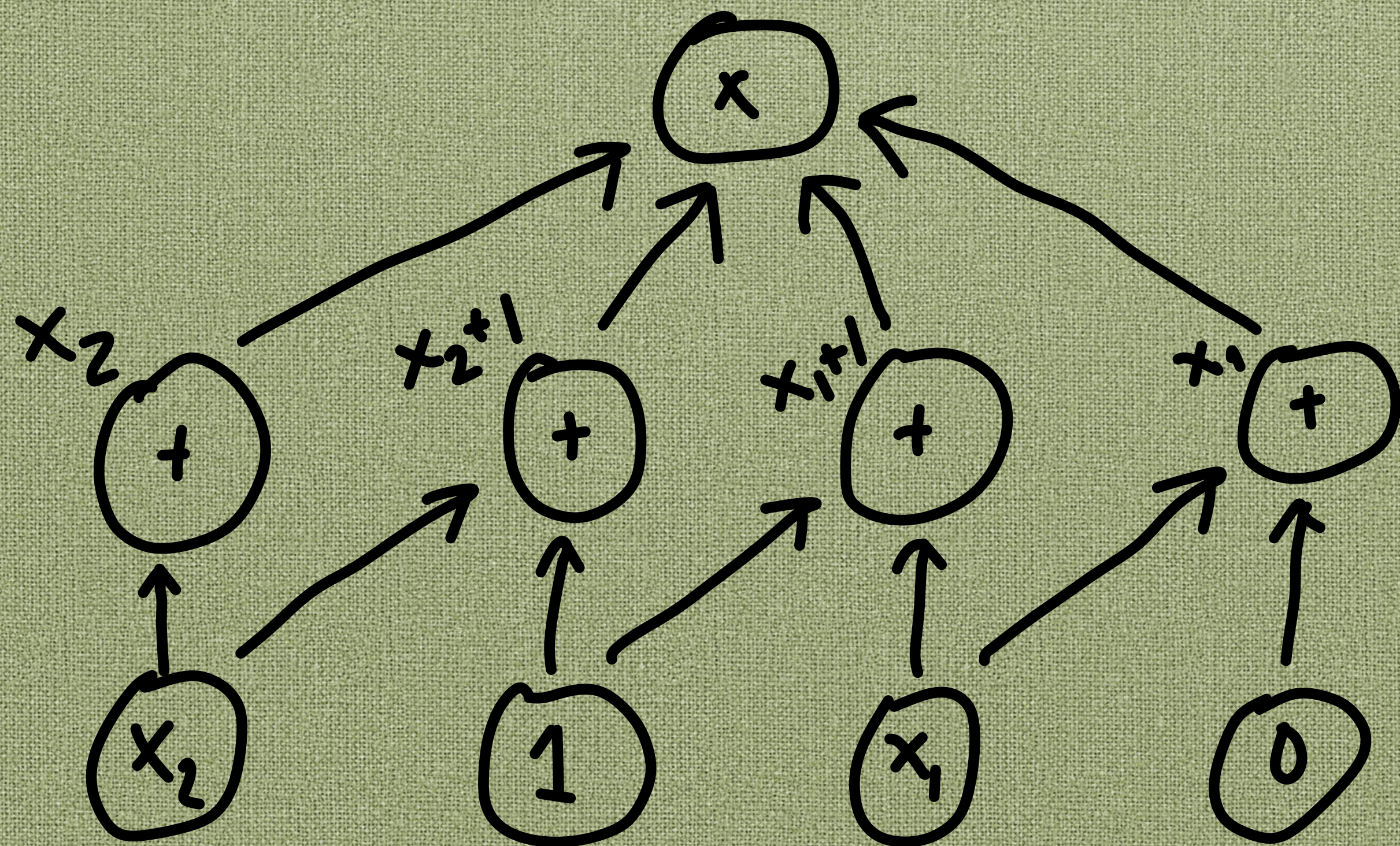
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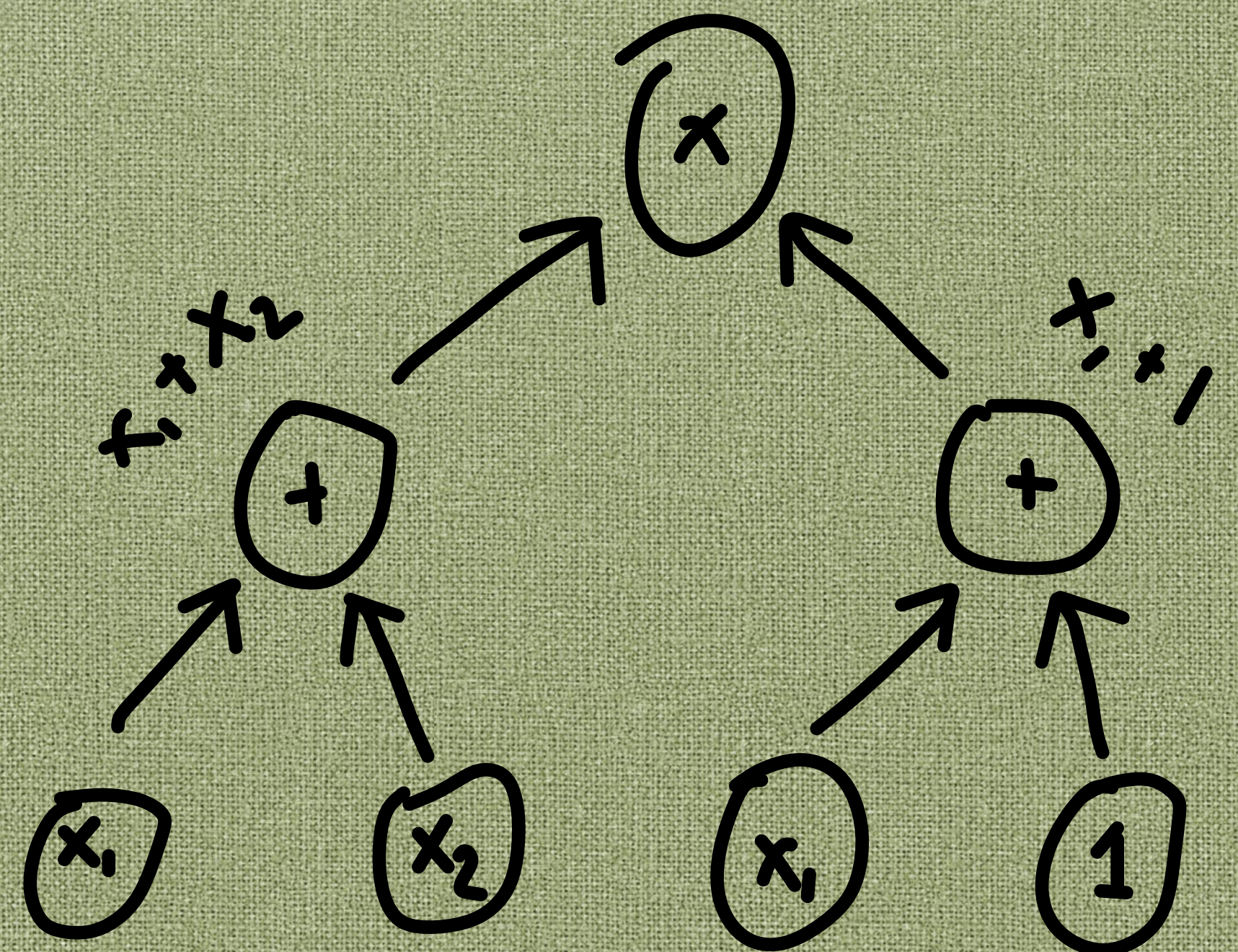


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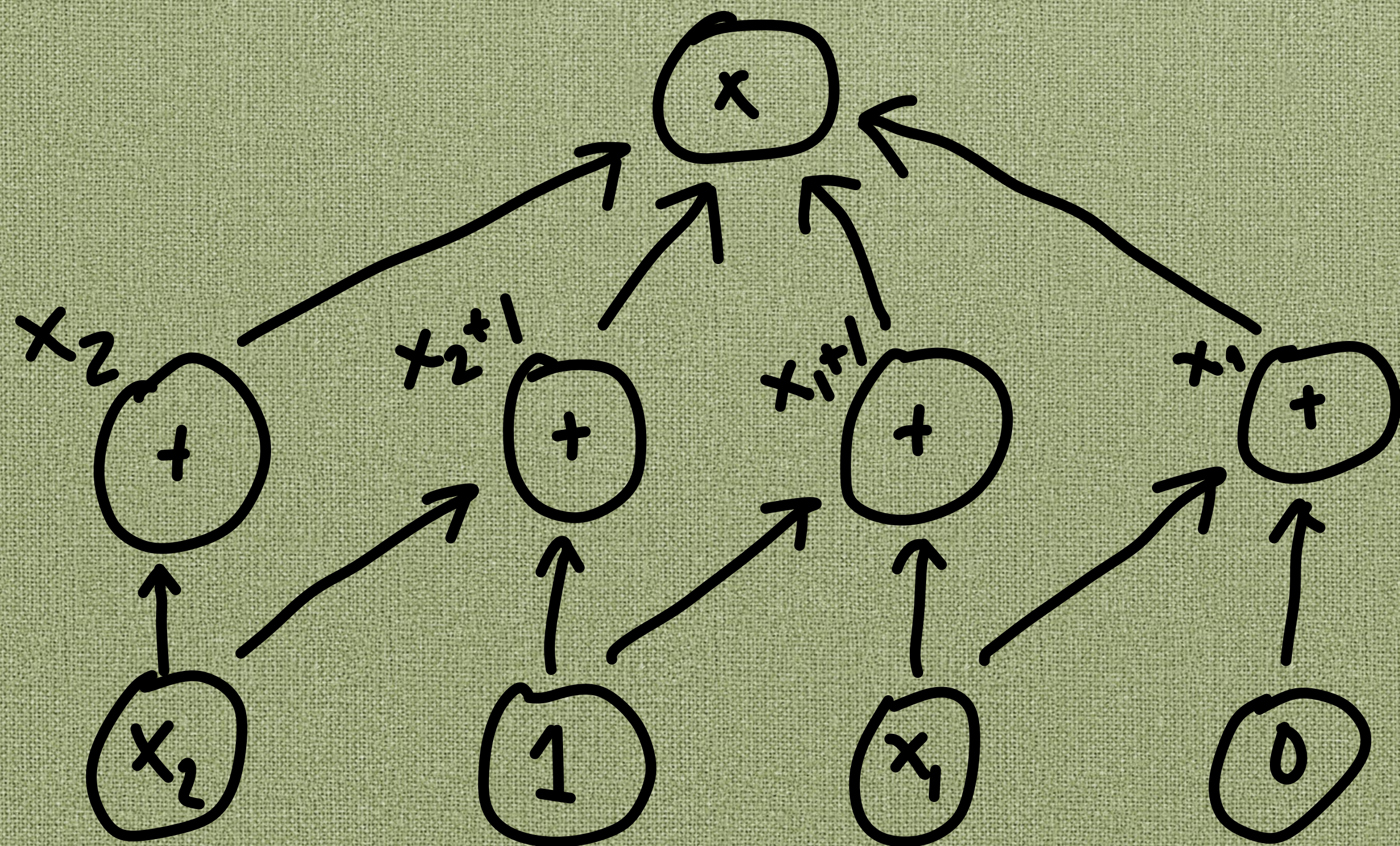
$$(x_1 + x_2) \cdot (x_1 + 1)$$



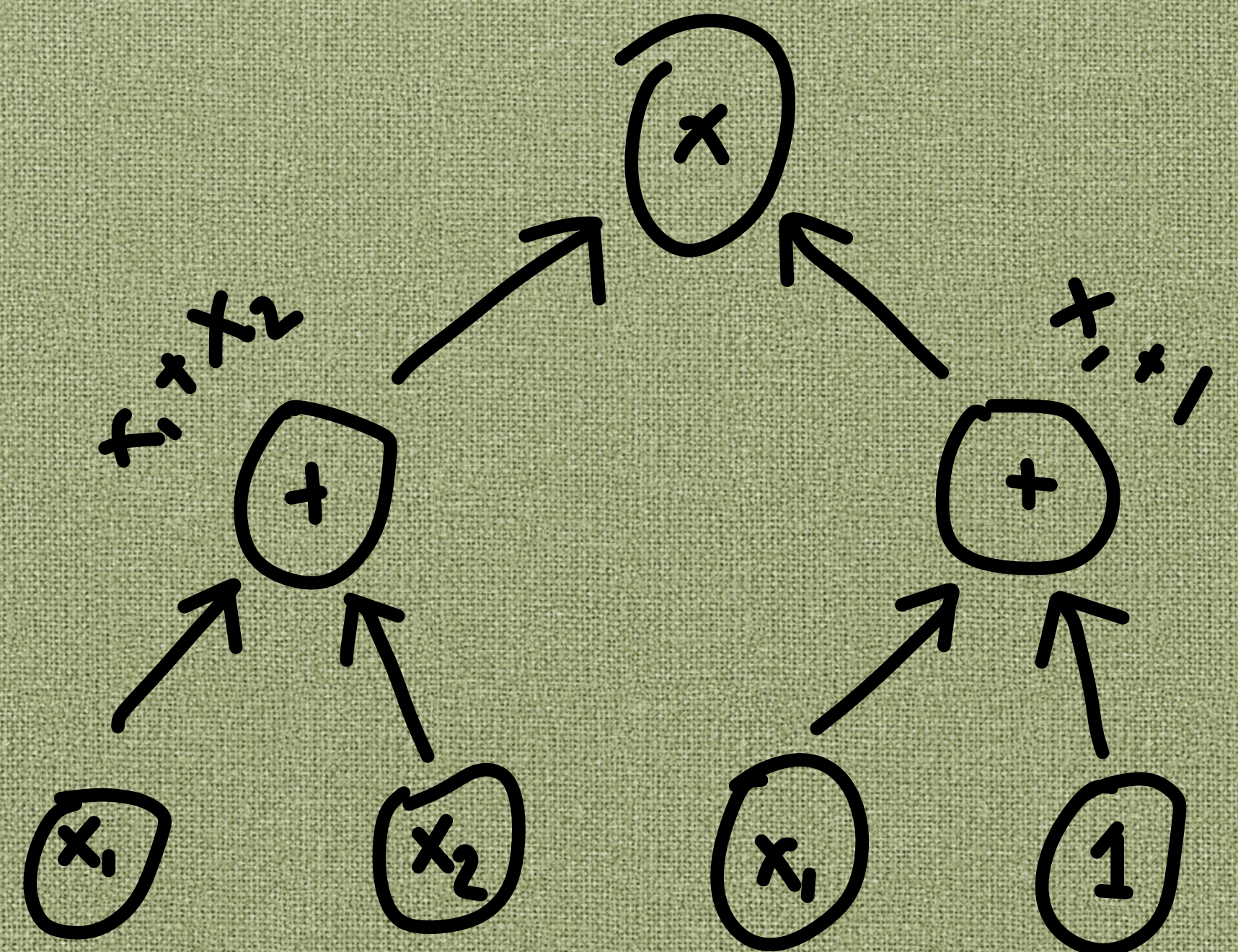
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VP-poly circuits

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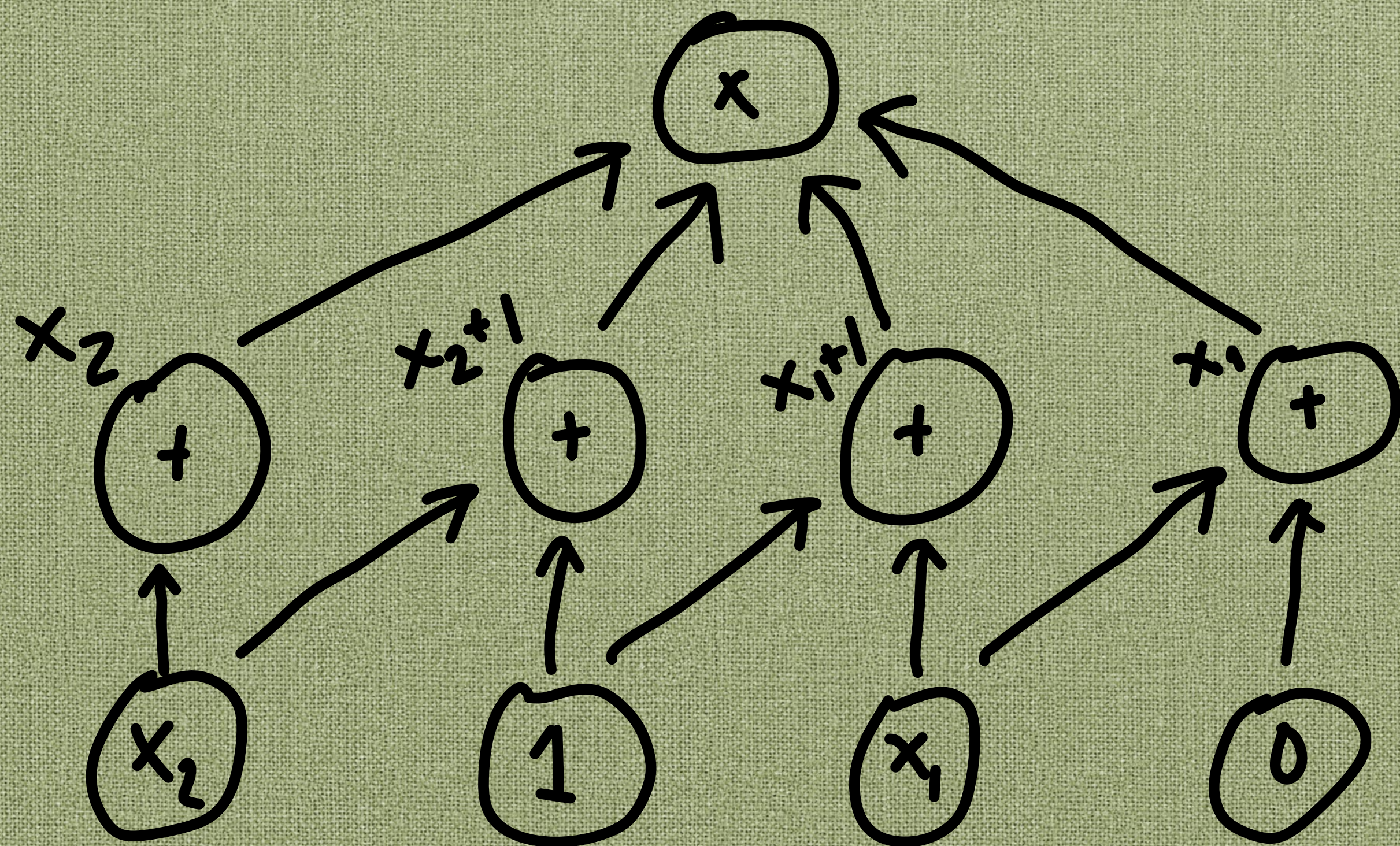
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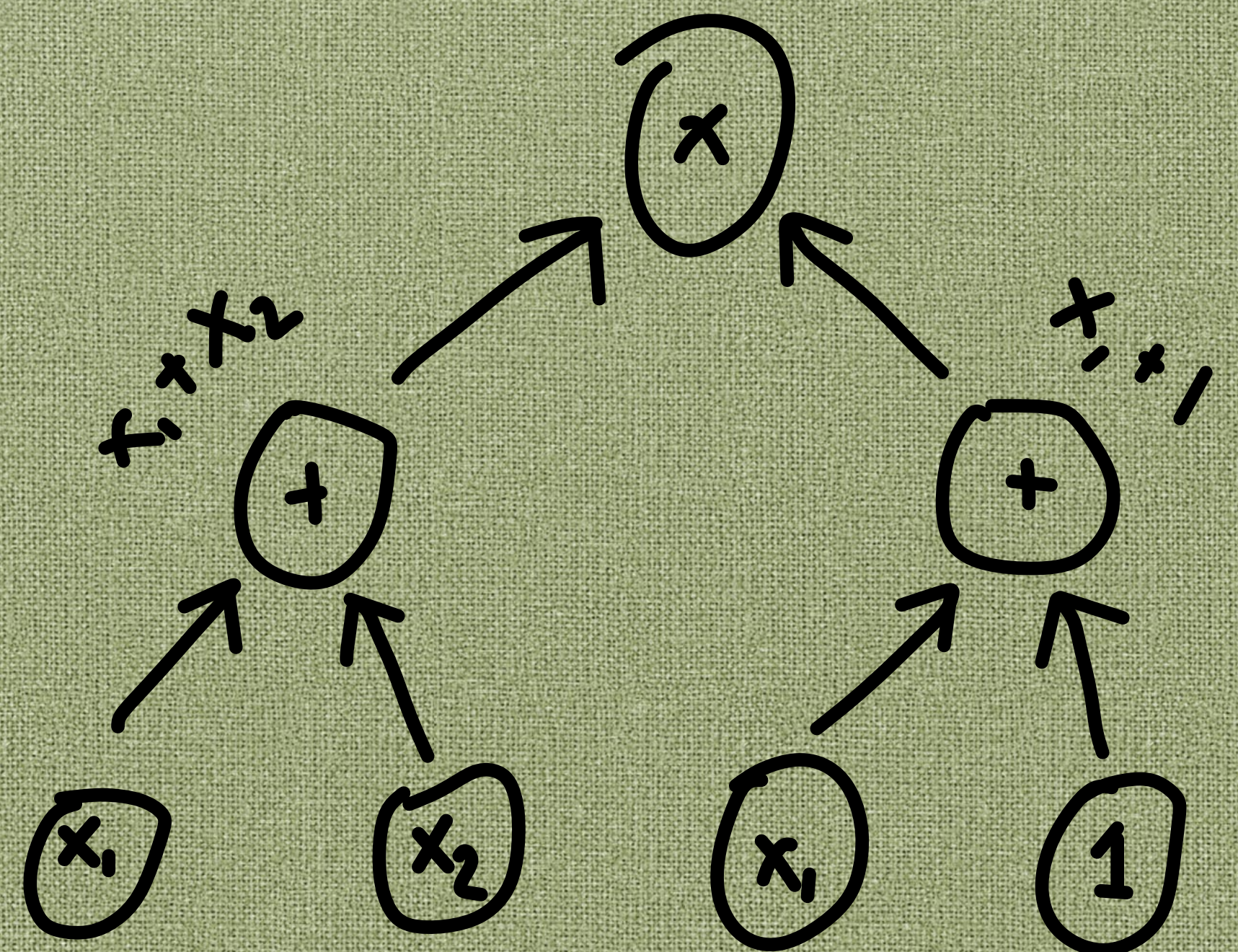
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VF-poly formulas

$$(x_1 + x_2) \cdot (x_1 + 1)$$



Algebraic Complexity

Projections:

$$f(x_1, x_2, x_3, x_4) = x_1 x_2 x_3 + x_1 x_4 + x_4^2$$

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x_1	$\leftarrow$	y_1
x_2	$\leftarrow$	y_1
x_3	$\leftarrow$	2
x_4	$\leftarrow$	y_2

Algebraic Complexity

Projections:

$$f(x_1, x_2, x_3, x_4) = x_1 x_2 x_3 + x_1 x_4 + x_4^2$$



$x_1 \leftarrow y_1$
$x_2 \leftarrow y_1$
$x_3 \leftarrow 2$
$x_4 \leftarrow y_2$

$$g(y_1, y_2) = 2y_1^2 + y_1 y_2 + y_2^2$$

Algebraic Complexity

(f_n) reduces to (g_n) :

$\forall n$, project $f_n \rightarrow g_n$

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(g_n) is $\mathcal{L}$ -complete:

- $\forall (f_n) \in \mathcal{L}$, (f_n) reduces to (g_n)

- $(g_n) \in \mathcal{L}$

Algebraic Complexity

Boolean Sum:

Let $S \subseteq \{x_1, \dots, x_n\}$,

$$\sum_S f = f|_{S \leftarrow 0} + f|_{S \leftarrow 1}$$

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$$\sum_S f = f|_{S \leftarrow 0} + f|_{S \leftarrow 1}$$

Ex: $S = \{x_1, x_2\}$

$$f = x_1 x_2 x_3 + x_1 x_4 + x_4^2$$

$$\begin{aligned} \sum_S f &= 0 \cdot x_3 + 0 \cdot x_4 + x_4^2 + x_3 + x_4 + x_4^2 \\ &= 2x_4^2 + x_3 + x_4 \end{aligned}$$

Algebraic Complexity

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★ Note: Usually defined with $|S_i| = 1$

Algebraic Complexity

$$\det_n = \sum_{\sigma \in S_n} (-1)^{\text{sgn}(\sigma)} \prod_{i=1}^n x_{i, \sigma(i)}$$

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$$

Algebraic Complexity

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$\det_n$	per_n
VP	VNP-complete

What is special about
the permanent?

Proving VNP-Completeness

$$f_n = \sum_{s_1} \cdots \sum_{s_e} g_m$$

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per simulates formulas

$$f_n \leq \sum_{s_1} \cdots \sum_{s_e} \text{per}_m$$

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our result

Assume $|s_i| = 2 \quad \forall i \in [\ell]$

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study h_{per} , smallest function st

$$\sum_s \text{per}_m \leq \text{per}_{h_{\text{per}}(m)}$$

Proving VNP-Completeness

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$\Downarrow$

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Want: $\underbrace{h_f(\dots h_f(m)\dots)}_{\text{poly times}}$ be a poly

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Boolean Sum Complexity

$$h_{\text{per}}(m) \leq m+3, \quad h_{\text{det}}(m) \geq 2(m-2)$$

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Boolean Sum Complexity

$$h_{\text{per}}(m) \leq m+3,$$

$$h_{\text{det}}(m) \geq 2(m-2)$$

$$\underbrace{h_{\text{per}}(\dots h_{\text{per}}(m)\dots)}_{k \text{ times}} \leq m+3k$$

$$\underbrace{h_{\text{det}}(\dots h_{\text{det}}(m)\dots)}_{k \text{ times}} \geq 2^k(m-4)$$

Worst Case

We will focus on

$$f = \sum_{\{x_{11}, x_{22}\}} \det_n$$

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$$M(\gamma) = \begin{pmatrix} \gamma & x_{12} & x_{13} \\ x_{21} & \gamma & x_{23} & \dots \\ x_{31} & x_{32} & x_{33} & \dots \\ \vdots & & & \ddots \end{pmatrix}, \quad f(x) = \det(M(0)) + \det(M(1))$$

Observe that $h_{\det}(n) = dc(f)$

Mignon - Ressaye

Hessien:

$$H_f(a) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(a) \right)_{i,j \in [n]}$$

Mignon - Ressaye

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Consider $\text{rank } H_f(a)$ when $f(a) = 0$

Thm: If $f(a) = 0$,

$$\dim \mathcal{L}(f) \geq \frac{1}{2} \text{rank } H_f(a)$$

Our Lower Bound

Recall: $f = \sum_{\{x_{11}, x_{22}\}} \det_n$

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Pick: $A = \begin{pmatrix} \gamma & 1 & & 0 \\ 1/2 & \gamma & & 0 \\ 0 & & & I_{n-2} \end{pmatrix}$

Our Lower Bound

Recall: $f = \sum_{\{x_{11}, x_{22}\}} \det_n$

Want $A \in \mathbb{F}^{n \times n}$ st $f(A) = 0$ For "non-trivial" reasons

Pick: $A = \begin{pmatrix} \gamma & 1 & 0 \\ 1/2 & \gamma & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix}$

$$f(A) = \det \begin{pmatrix} 0 & 1 \\ 1/2 & 0 \end{pmatrix} + \det \begin{pmatrix} 1 & 1 \\ 1/2 & 1 \end{pmatrix} = -\frac{1}{2} + \frac{1}{2} = 0$$

Our Lower Bound

$$\frac{\gamma^2}{\gamma x_{ij} \gamma x_{kl}}$$

$\equiv$

delete rows i, k
+ cols j, l

Our Lower Bound

$$\frac{\partial^2}{\partial x_{ij} \partial x_{kl}} \equiv \text{delete rows } i, k \\ + \text{cols } j, l$$

For example:

$$\frac{\partial^2}{\partial x_{12} \partial x_{21}} \equiv$$

$$A = \begin{pmatrix} \cancel{x} & \cancel{1} & \cancel{0} \\ \cancel{1} & \cancel{x} & \cancel{0} \\ 0 & 0 & I_{n-2} \end{pmatrix}$$

Our Lower Bound

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For example:

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$$A = \begin{pmatrix} x & 1 & 0 \\ 1/2 & y & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix}$$

$$\frac{\partial^2}{\partial x_{1i} \partial x_{i1}} \equiv$$

$$A = \begin{pmatrix} x & 1 & 0 \\ 1/2 & y & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix}$$

Our Lower Bound

Pick variables

$$x_{i1}, x_{1i}, x_{i2}, x_{2i}$$

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We prove: $\text{rank } H_f(A) \geq 4(n-2)$

Conclusion

To show (f_n) VNP-complete, study

$$\sum_{s_1} \cdots \sum_{s_q} f_n$$

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Proof idea: Iteratively combine

$$\sum_{s_1} f_n = f_{h_f(n)}, \quad \underbrace{h_f(\cdots h_f(n) \cdots)}_{n^c \text{ times}} = n^{c'}$$

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Permanent

Determinant

$$h_{\text{per}}(n) \leq n+3$$

$$h_{\text{det}}(n) \geq 2n-4$$

Thank You!

Boolean Sum Lower Bound

For $i > 2$:

Study $x_{1i}, x_{2i}, x_{i1}, x_{i2}$

$$\begin{matrix} & x_{1i} & x_{2i} & x_{i1} & x_{i2} \\ \begin{matrix} x_{1i} \\ x_{2i} \\ x_{i1} \\ x_{i2} \end{matrix} & \begin{pmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & 2 & -1 \\ -1 & 2 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix} \end{matrix}$$

Permanent Upper Bound

$$U = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ -\frac{1}{2} & \frac{1}{2} & \frac{3}{2} \end{pmatrix}$$

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Permanent Upper Bound

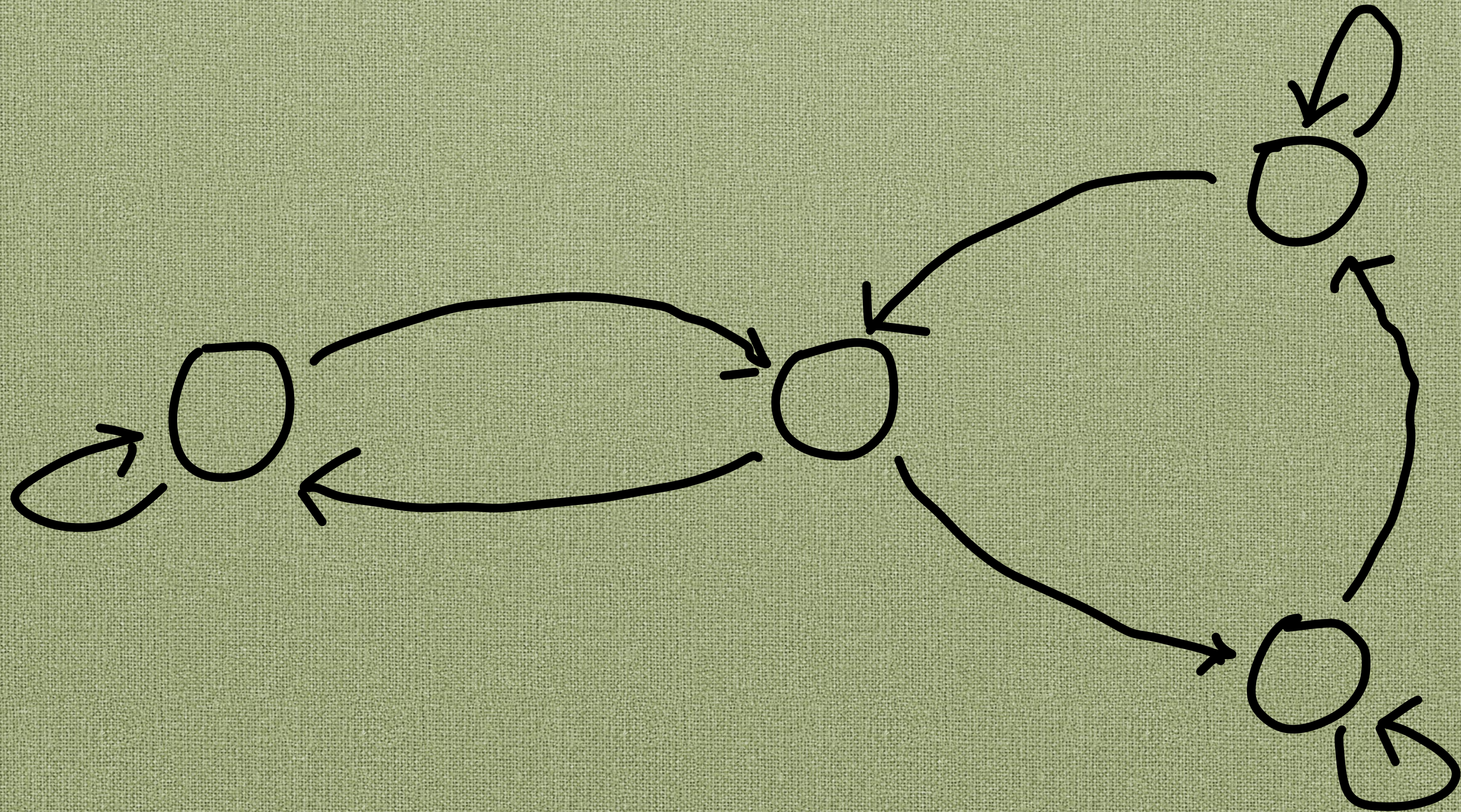
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Thus, $h_{\text{per}}(n) \leq n+3$.

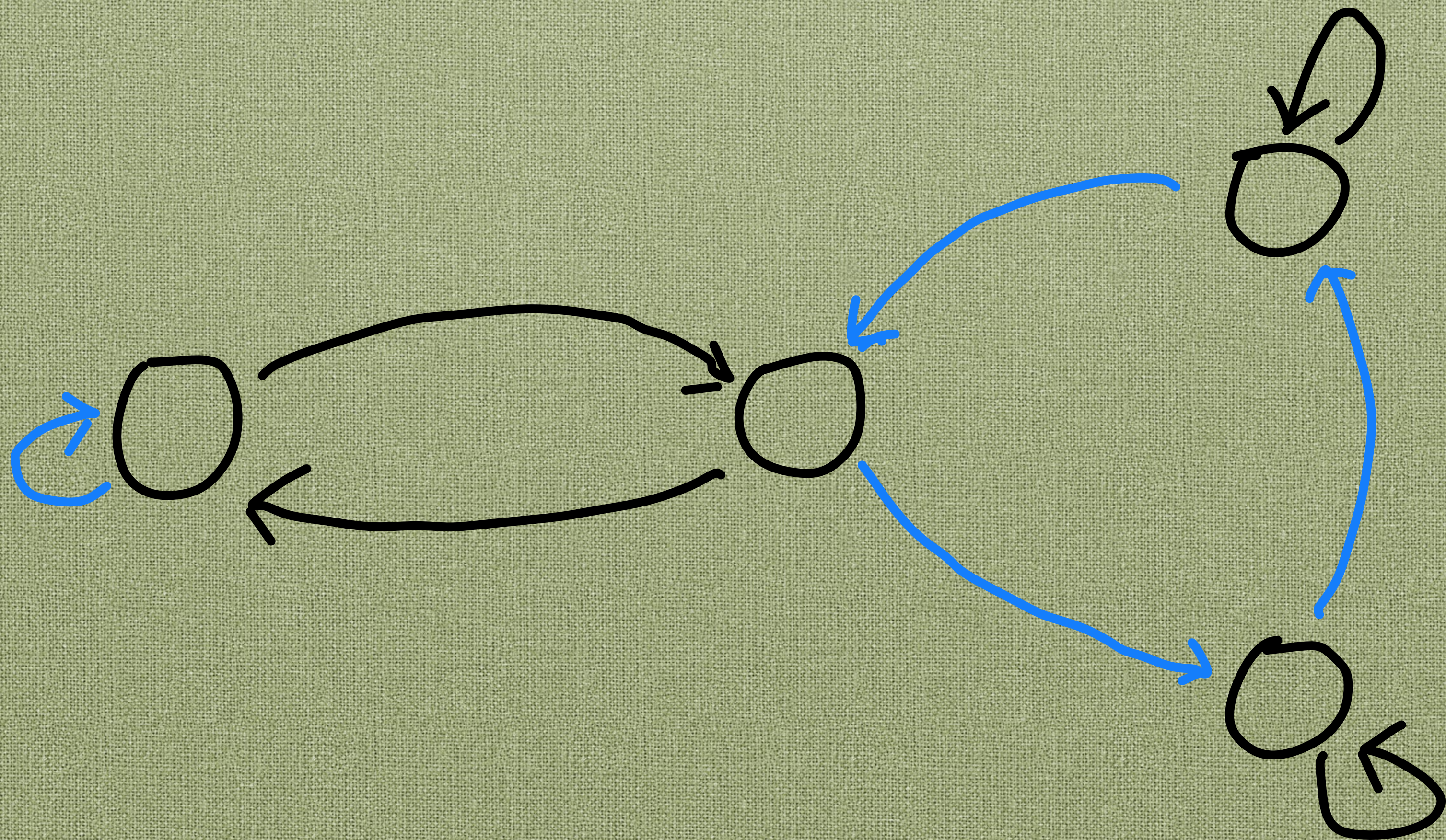
Per/Det $\equiv$ Cycle covers

Cycle cover



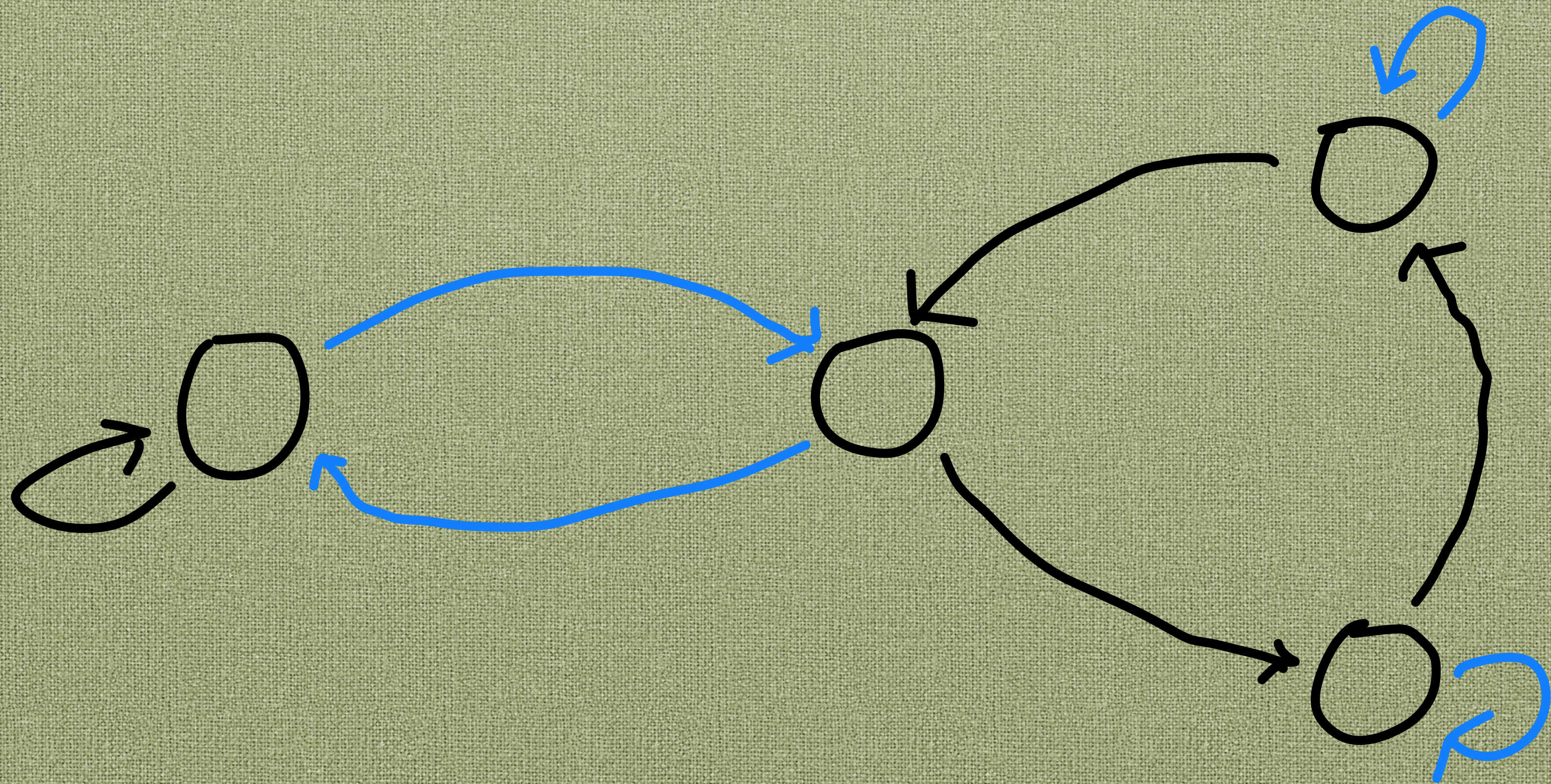
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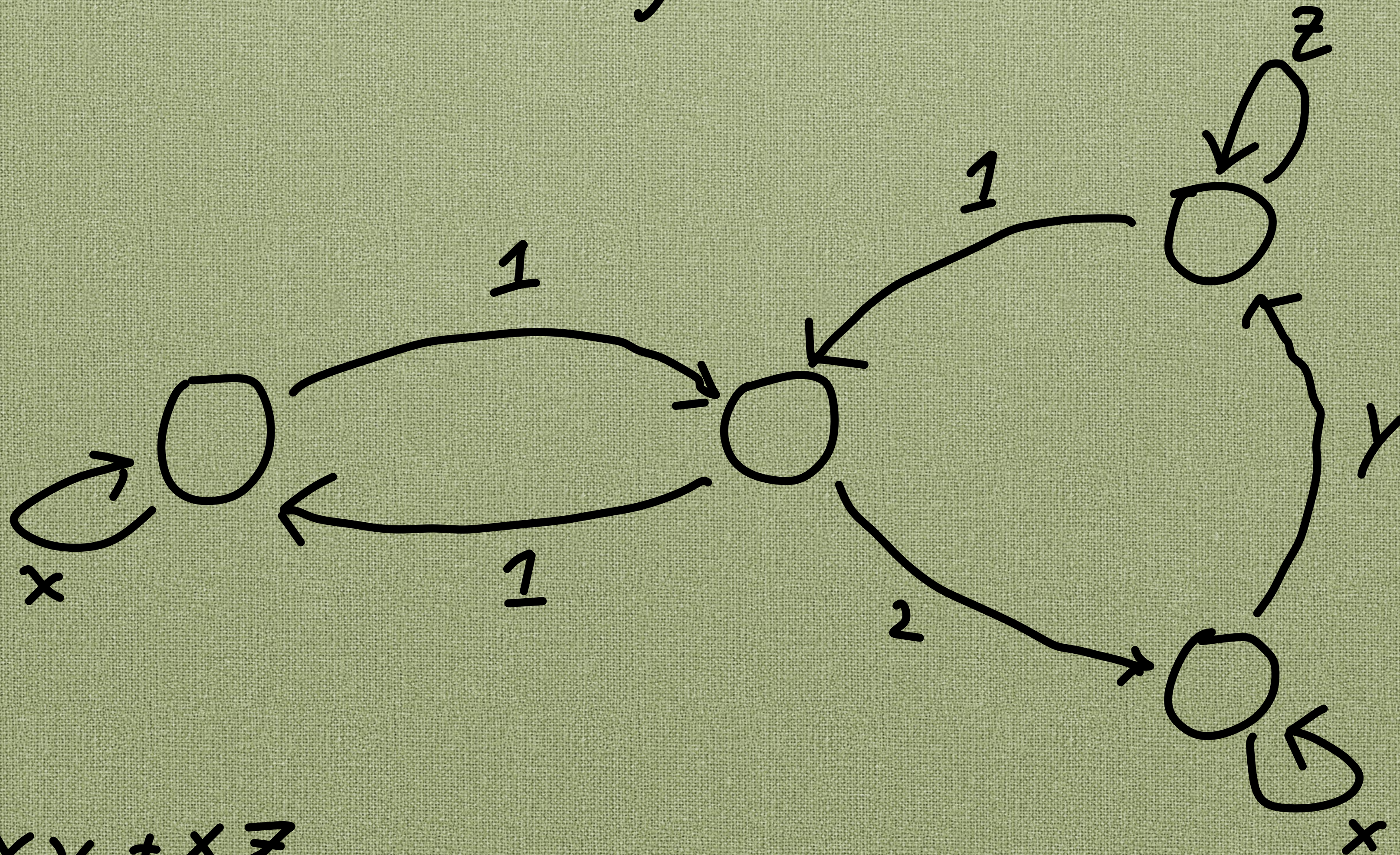
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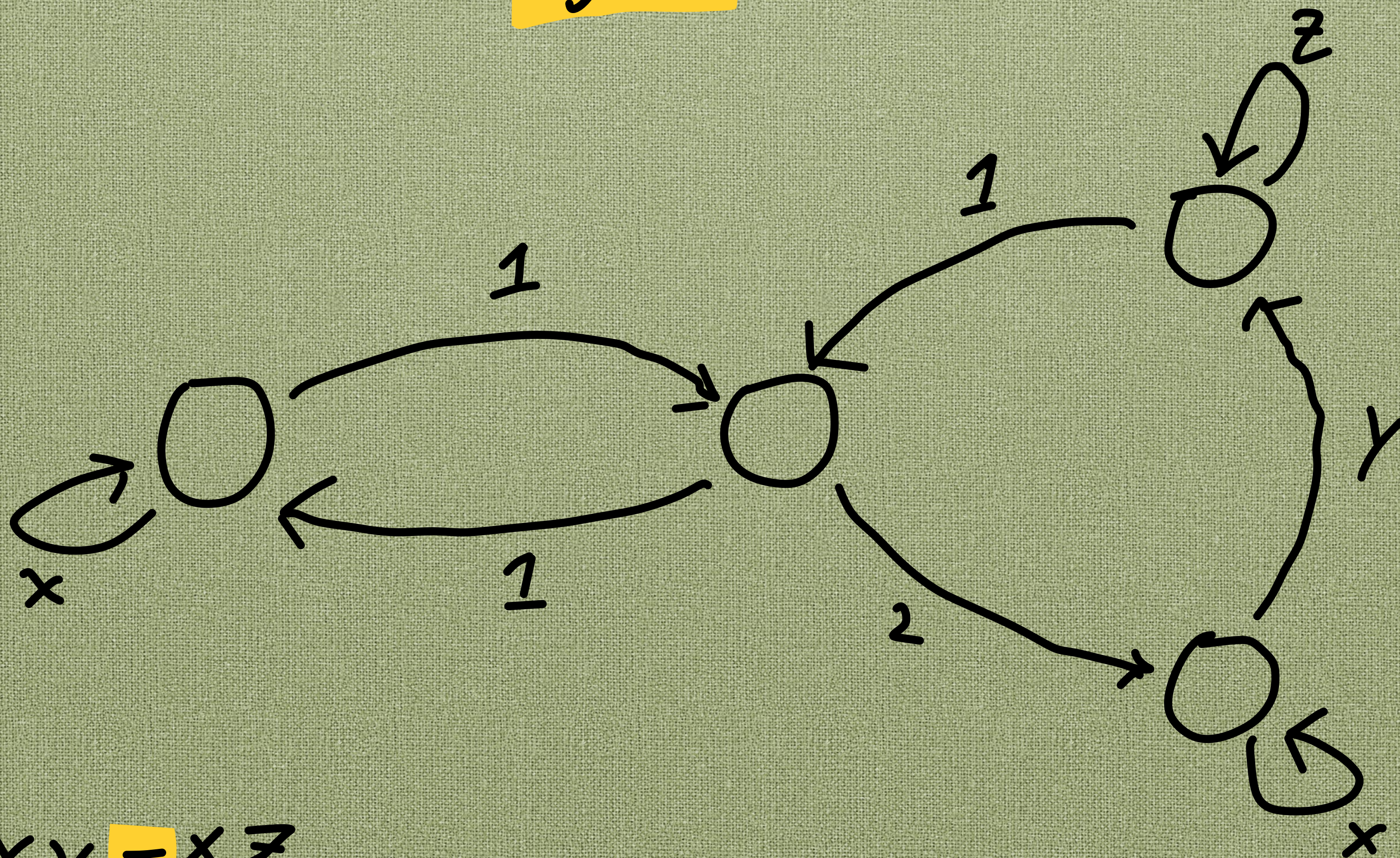
Per = sum of weighted cycle covers



$$2xy + xz$$

Per/Det $\equiv$ Cycle covers

det = sum of **signed**, weighted cycle covers



$$2xy - xz$$

Reduction to sets of size 2

$$\text{Let } S = \{x_1, \dots, x_k\}$$

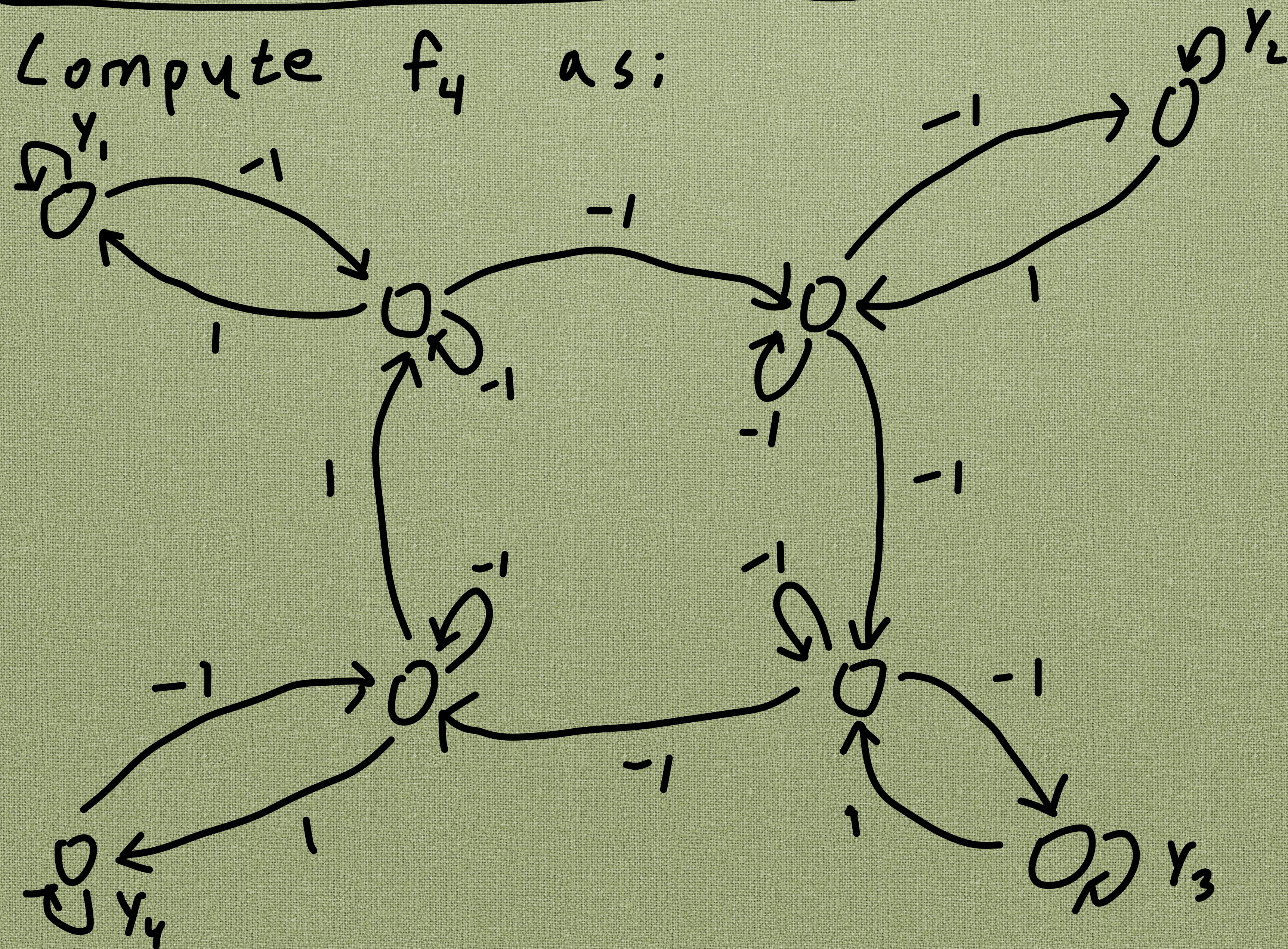
$$\text{Let } f_k = \prod_{i=1}^k y_i + \prod_{i=1}^k (1 - y_i)$$

Thus,

$$\sum_S \det_n = \sum_{\{x_1, y_1\}} \cdots \sum_{\{x_k, y_k\}} (f_k \cdot \det_n)$$

Reduction to sets of size 2

Compute f_4 as:



Example

$$A = \begin{pmatrix} 0 & 0 \\ 0 & I_{n-1} \end{pmatrix} \in \mathbb{F}^n, \quad \det(A) = 0$$

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$$H_{\det}(A) = \begin{pmatrix} B & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathbb{F}^{n^2 \times n^2}$$

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$$B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \ddots & 1 \end{pmatrix} \in \mathbb{F}^{(2n-1) \times (2n-1)}$$

over $x_{i,1}, x_{i,i} \forall i \in [n]$

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over $x_{i,1}, \dots, x_{i,i} \quad \forall i \in [n]$

$$C = \begin{pmatrix} 0 & 1 & \dots & 1 \\ \vdots & & & 0 \end{pmatrix} \in \mathbb{F}^{n \times n}$$

over $x_{1,1}, \dots, x_{n,n}$

Example

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over $x_{1,1}, \dots, x_{n,n}$

Thus, $\text{rank } H_{\det}(A) = 2n$

Mignon - Ressaye

Consider some $M = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ st $\det(M) = 0$
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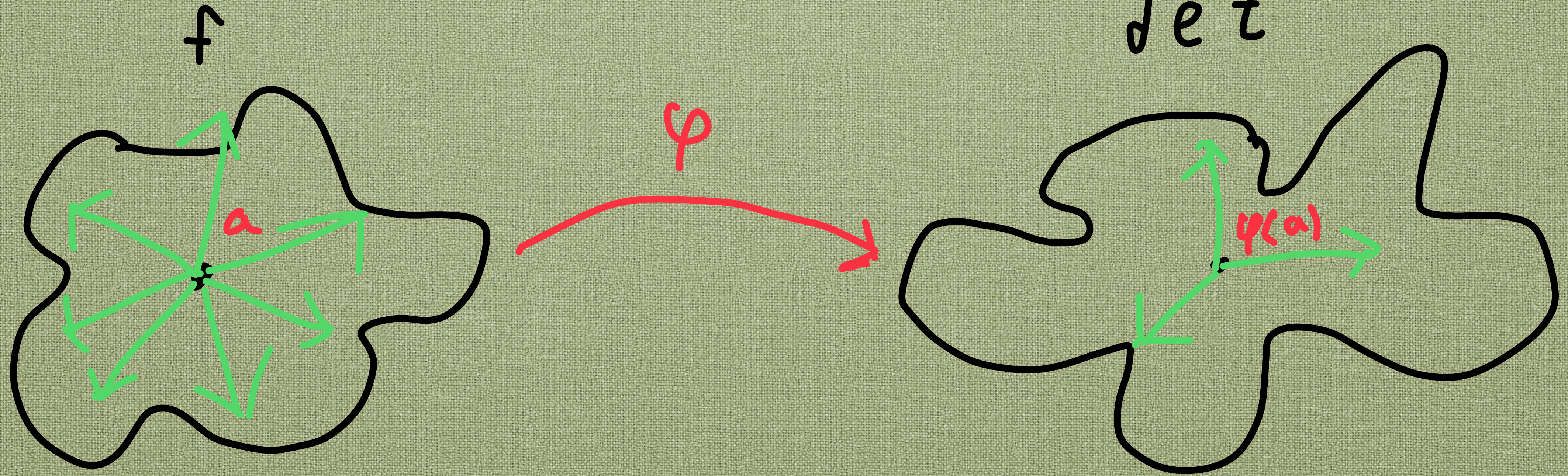
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Consider some $M = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ st $\det(M) = 0$
(say $\text{rank}(M) = n-1$)

Observe that $\det\left(M + \underbrace{\begin{pmatrix} \text{span}(v_1, \dots, v_n) \\ \vdots \\ \text{span}(v_1, \dots, v_n) \end{pmatrix}}_{\text{very large}}\right) = 0$

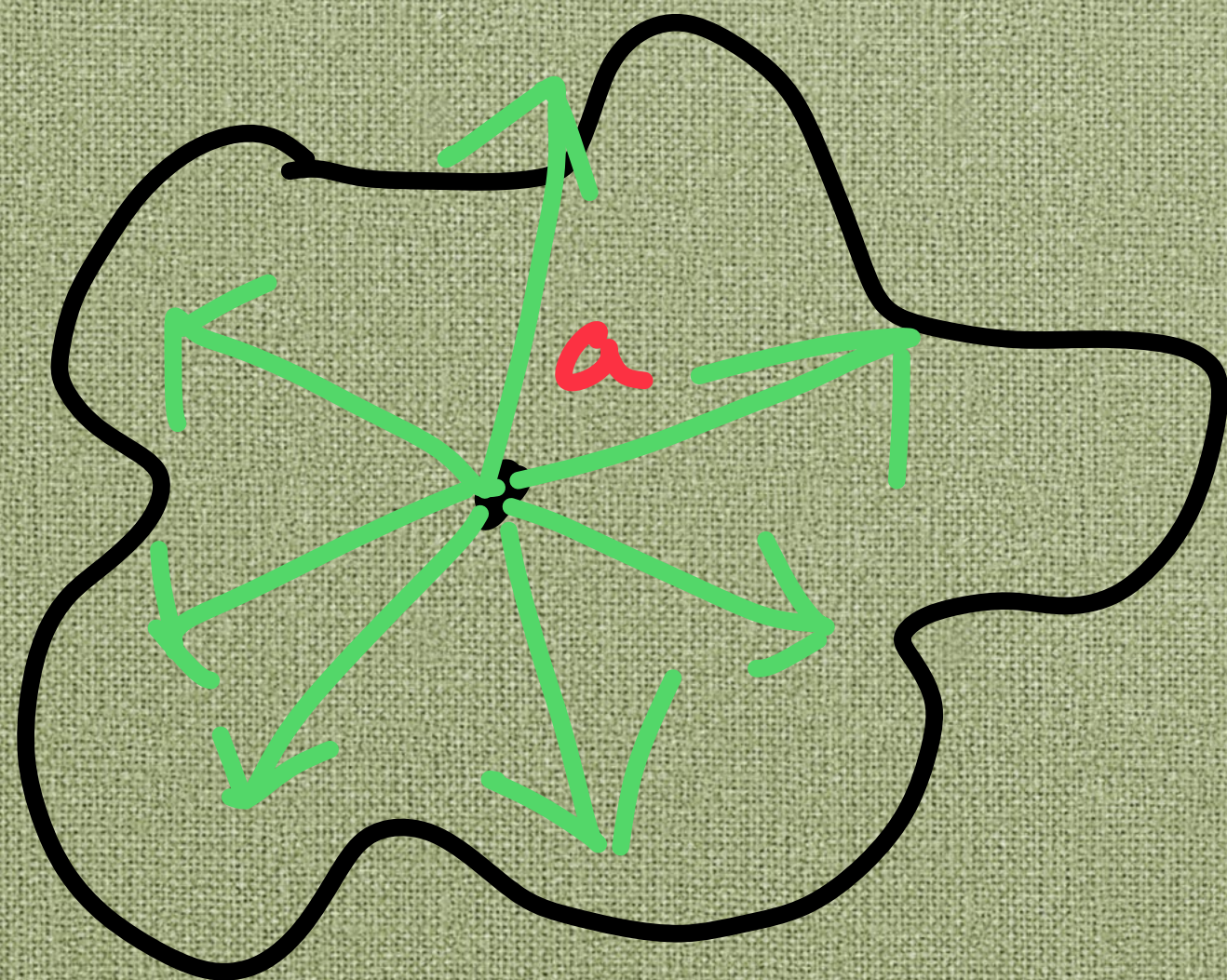
$$\dim \begin{pmatrix} \text{span}(v_1, \dots, v_n) \\ \vdots \\ \text{span}(v_1, \dots, v_n) \end{pmatrix} = n \cdot (n-1)$$

Mignon - Ressaye



Mignon - Ressarve

f

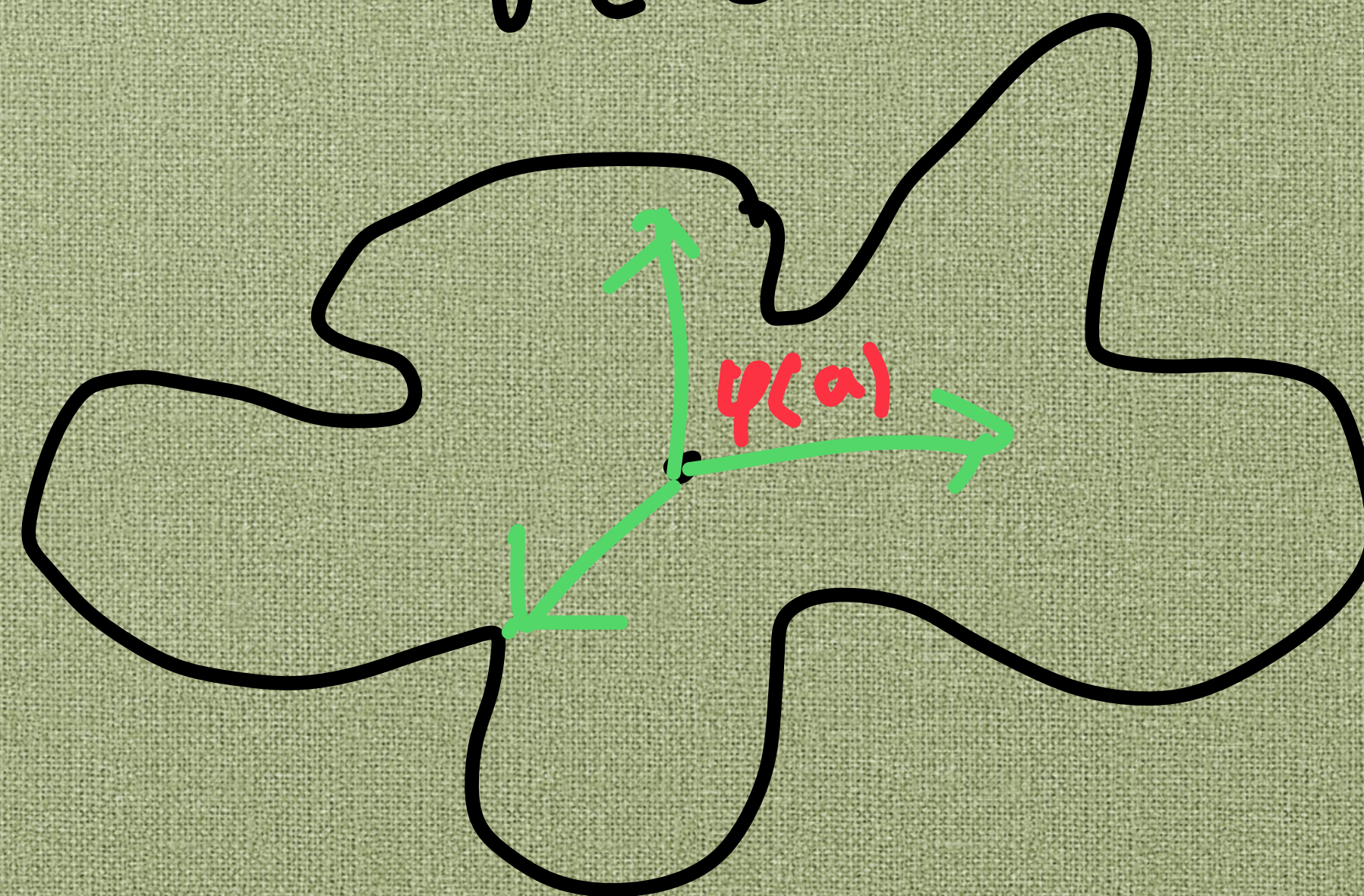


Many non-zero
directions

φ

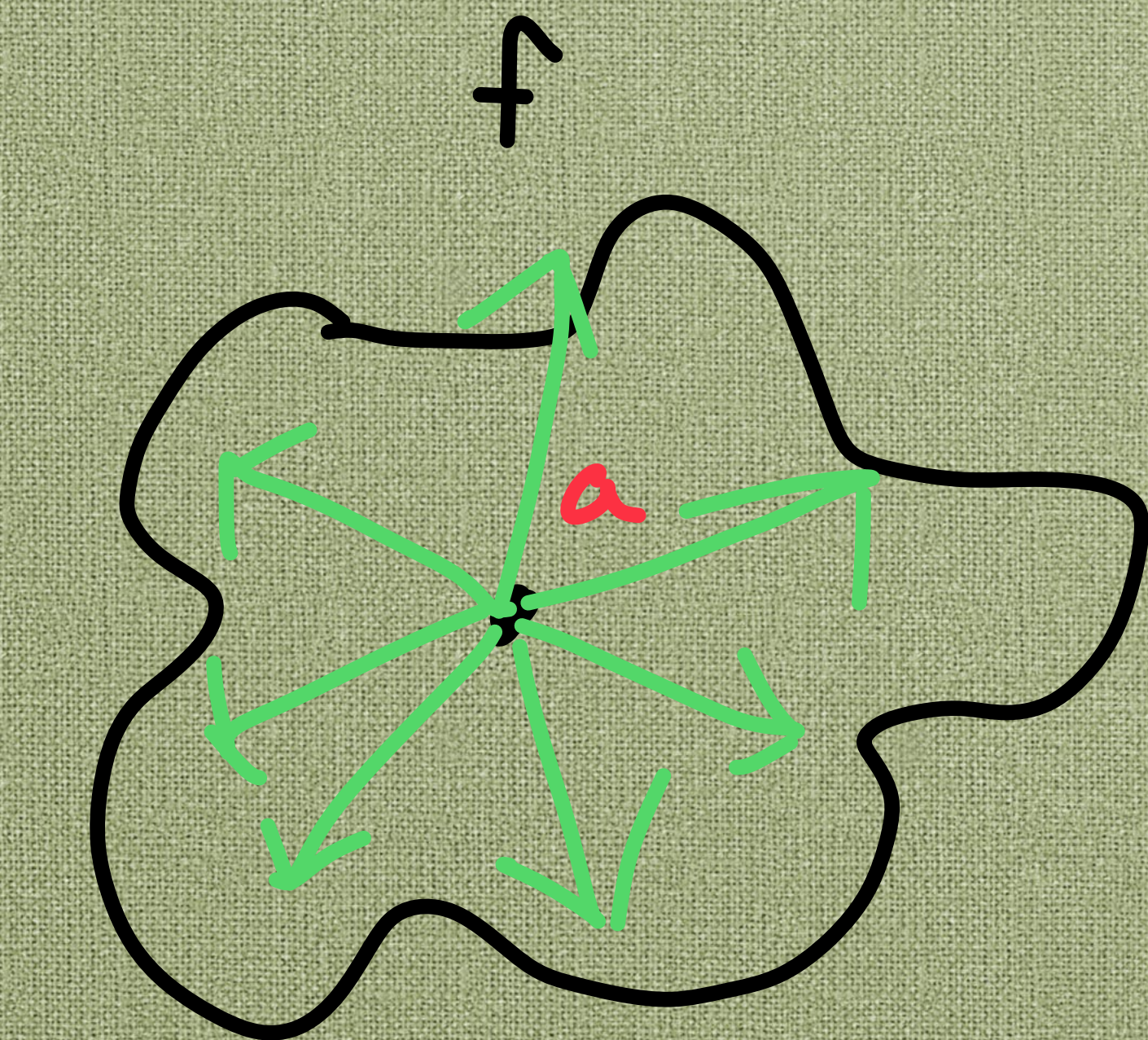


jet

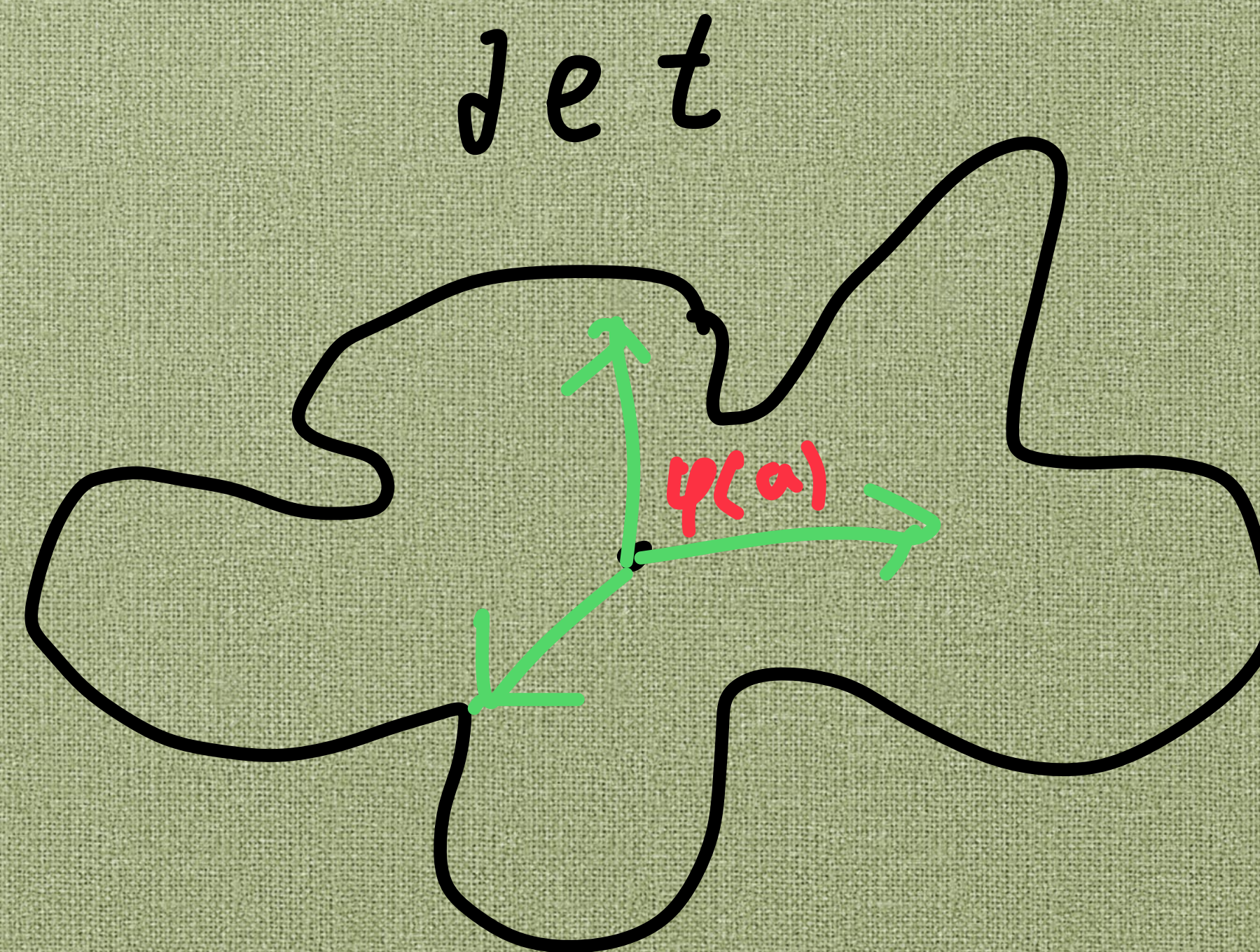
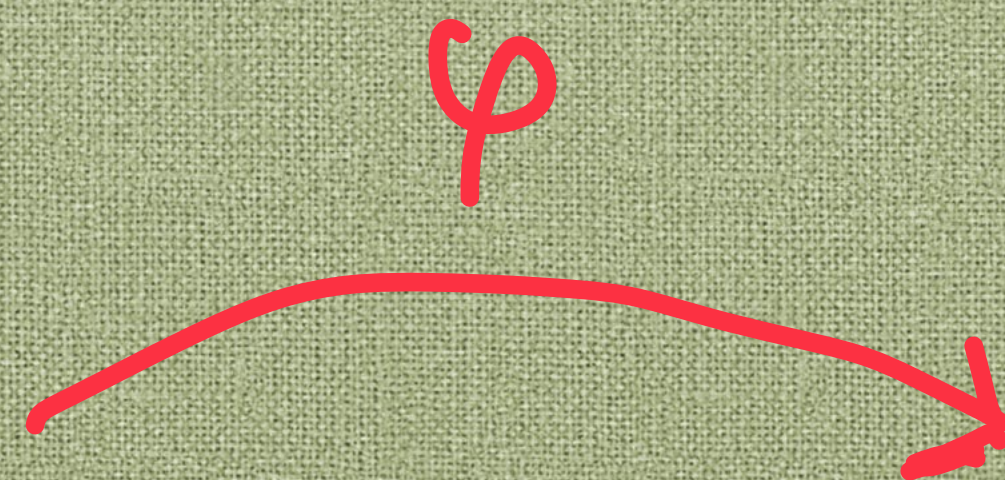


Few non-zero
directions

Mignon - Ressaye



Many non-zero
directions



Few non-zero
directions

Study the Hessian